NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS
POST GRADUATE

PG (MT) - X B(II)

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Mechanics of Solids

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of proper lay-out of the meterials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great part of these efforts is still experimental–in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these to admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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PGMT XB(II)
Mechanics of Solids

(Sp. paper : Applied Mathematics)

Unit 1 □ Two Dimensional Elastostatic Problems

Structure

1.1 Introduction

Here we study a general method of solution of certain broad classes of two-dimensional boundary-value problems is elasticity. The method is based on the reduction of the boundary-value problems in elasticity to the solutions of certain fundamental equations in a complex domain. Two-dimensional problem with which we shall be concerned can be classified into two physically distinct types. viz. plane strain and plane stress.

1.2 Plane strain

A body is said to be in the state of **plane strain** or **plane deformation** parallel to the x_1x_2 plane, if the component u_3 of the displacement vector $\vec{u} = (u_1, u_2, u_3)$ vanishes and the components u_1 and u_2 are functions of the coordinates x_1 and x_2, but not of x_3. Thus in the plane deformation we have

$$u_\alpha = u_\alpha(x_1, x_2),\ (\alpha = 1,2), \qquad\text{.........................(1.1)}$$

$$u_3 = 0$$

The strain and rotational components are

$$e_{\alpha\beta} = \tfrac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right),\ (\alpha,\beta = 1,2) \qquad\text{................(1.2)}$$

$$w_{\alpha\beta} = \tfrac{1}{2}\left(u_{\alpha,\beta} - u_{\beta,\alpha}\right).$$

The stress-strain relations for isotropic medium, in this case are

$$\left.\begin{aligned} &\tau_{\alpha\beta} = \lambda\vartheta\,\delta_{\alpha\beta} + \mu\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right).\ (\alpha,\beta = 1,2) \\ &\tau_{33} = \lambda\vartheta,\ \tau_{13} = \tau_{23} = 0. \\ &\text{where } \vartheta = u_{\alpha,\alpha} = u_{1,1} + u_{2,2} \end{aligned}\right\}\text{....(1,3).}$$

and λ, μ are Lame's constants.

From (1·3), we have

$$\tau_{11} + \tau_{22} = 2\lambda\vartheta + 2\mu\vartheta = 2(\lambda + \mu)\frac{\tau_{33}}{\lambda}\ (\text{by } 1{\cdot}3)$$

$$\text{i.e. } \tau_{33} = \sigma\left(\tau_{11} + \tau_{22}\right) \qquad (1{\cdot}4)$$

where $\sigma = \dfrac{\lambda}{2(\lambda + \mu)}$ is called the Poisson's ratio.

It follows from (1·2), (1·3) and (1·4) that the deformation and stresses in an elastic body in plane-strain condition are completely determined by the five quantities, viz. u_1, u_2 and τ_{11}, τ_{22}, τ_{12}.

Now it is known that the stress equations of equilibrium under body force F_i per unit volume are

$$\tau_{ij,j} + F_i = 0,\ (i, j = 1,2,3)$$

For : $i = j = 3$, this equation gives

$$\tau_{33,3} + F_3 = 0\,.$$

Since in the plane strain condition τ_{33} does not contain x_3, so $\tau_{33,3} = 0$ and, therefore, F_3=0.

Thus the above equilibrium equations contain only two equations given by

$$\tau_{\alpha\beta,\beta} + F_\alpha = 0,\ (\alpha,\beta = 1,2)\text{..............................(1·5)}$$

Substitution of (1·3) in (1·5) yields the Navier equations of equilibrium as

$$\mu\nabla_1^2 u_\alpha + (\lambda + \mu)\frac{\partial\vartheta}{\partial x_\alpha} = -F_\alpha\left(x_1, x_2\right)\text{................(1·6)}$$

where $\nabla_1^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$.

The equilibrium equations (1·6) are to be satisfied in some two-dimensional region R of the cross-section of the body formed by the plane x_3=constant. If C is the boundary of the region R, then the boundary conditions are to be satisfied on C.

The Beltrami-Michell compatibility equations, in this case, contain only one equation

$$\nabla_1^2 \Theta_1 = -\frac{2(\lambda+\mu)}{\lambda+2\mu} F_{\alpha,\alpha}, \ (\alpha=1,2) \text{ ....(1·7)}$$

where $\Theta_1 = \tau_{11} + \tau_{22}$.

The boundary conditions, when the stresses $T_\alpha(x_1, x_2)$ are prescribed along the contour C are

$$\tau_{\alpha\beta} \nu_\beta = T_\alpha(x_1, x_2), \ (\alpha=1,2), \text{ ....(1·8)}$$

where ν_α are components of exterior unit normal vector on C.

We seek a solution of the system of equations (1·5) and (1·7) in the region R subject to the condition (1·8) on the boundary C.

1.3 Plane Stress and Generalized Plane Stress

Plane Stress :

A body is said to be in the state of **plane stress** parallel to the x_1 x_2 plane when the stress components $\tau_{13}, \tau_{23}, \tau_{33}$ vanish.

From the stress-strain relations,

$$\tau_{ij} = \lambda\vartheta\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \quad \vartheta = u_{i,i} \ (i,j=1,2,3) \text{ ....(1·9)}$$

we have

$$\tau_{33} = \lambda\vartheta + 2\mu u_{3,3}.$$

Since for plane-stress condition parallel to x_1 x_2 plane

$\tau_{33} = 0$, so

$$\lambda(u_{1,1} + u_{2,2} + u_{3,3}) + 2\mu\, u_{3,3} = 0.$$

or, $$u_{3,3} = -\frac{\lambda}{\lambda+2\mu}(u_{1,1} + u_{2,2}) \text{ ....(1·10)}$$

Hence $$\vartheta = u_{1,1} + u_{2,2} + u_{3,3} = \frac{2\mu}{\lambda+2\mu}(u_{1,1} + u_{2,2}) = \frac{2\mu}{\lambda+2\mu}\vartheta_1$$

where $\vartheta_1 = u_{1,1} + u_{2,2}$

Thus $\tau_{11} = \lambda\vartheta + 2\mu\, u_{1,1}$ reduces to

$$\left.\begin{aligned} \tau_{11} &= \frac{2\lambda\mu}{\lambda+2\mu}\vartheta_1 + 2\mu\, u_{1,1} \\ &\text{Similarly,} \\ \tau_{22} &= \frac{2\lambda\mu}{\lambda+2\mu}\vartheta_1 + 2\mu\, u_{2,2} \\ \text{and } \tau_{12} &= \mu\left(u_{1,2} + u_{2,1}\right) \end{aligned}\right\} \qquad \text{.........................(1·11)}$$

Substituting the stress components from (1·11) into the equation of equilibrium we get

$$\left(\frac{2\lambda\mu}{\lambda+2\mu} + \mu\right)\frac{\partial\vartheta_1}{\partial x_\alpha} + \mu\nabla_1^2 u_\alpha = -F_\alpha \qquad \text{.....................(1·12)}$$

where $\vartheta_1 = u_{1,1} + u_{2,2}$ and $\nabla_1^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$.

It is to be noted that in plane strain problem, the displacements u_α and stresses $\tau_{\alpha\beta}$ are independent of x_3-coordinate, whereas in the problem of plane stress, these functions may depend on x_3. However, the equations (1·6) and (1·12) become identical, if one replaces the constant $\dfrac{2\lambda\mu}{\lambda+2\mu} = \bar{\lambda}$ by λ.

Generalized plane stress :

Consider a cylinder with the generators parallel to the x_3-axis and with bases in the planes $x_3 = \pm h$, the height $2h$ being very small compared to the linear dimensions of the cross-section. Now for thin plates the mean values $\bar{u}_i$ of the displacements u_i give useful information as that furnished by the u_i. This suggests dealing with the average values

$$\bar{u}_i(x_1, x_2) \equiv \frac{1}{2h}\int_{-h}^{h} u_i(x_1, x_2, x_3)\,dx_3, \qquad \text{...............(1·13)}$$

where $\bar{u}_3 = 0$.

Since the faces of the plates are assumed to be free of external loads, so

$$\tau_{13}(x_1, x_2, \pm h) = \tau_{23}(x_1, x_2, \pm h) = \tau_{33}(x_1, x_2, \pm h) = 0 \qquad \text{........(1·14)}$$

and these equations together with the equation of equilibrium

$$\tau_{13,1}+\tau_{23,2}+\tau_{33,3}=0$$

require $\tau_{33,3}(x_1,x_2,\pm h)=0$. Since τ_{33} and its derivative with respect to x_3 vanish on the faces of the plate, so τ_{33} can differ from zero but slightly throughout the plate if h is small. The stress plate will be said to be in a state of **generalized plane stress** if the condition.

$$\tau_{33}=0 \tag{1·15}$$

holds everywhere.

The remaining equations of equilibrium

$$\tau_{\alpha 1,1}+\tau_{\alpha 2,2}+\tau_{\alpha 3,3}+F_\alpha=0,\ (\alpha=1,2)$$

upon integration with respect to x_3 between the limits $-h$ and $+h$ and $+h$ give

$$\frac{1}{2h}\int_{-h}^{h}\left(\tau_{\alpha 1,1}+\tau_{\alpha 2,2}+\tau_{\alpha 3,3}+F_\alpha\right)dx_3=0,\ (\alpha=1,2) \tag{1·16}$$

$$\text{i.e., } \bar{\tau}_{\alpha 1,1}+\bar{\tau}_{\alpha 2,2}+\bar{F}_\alpha=0 \tag{1·17}$$

$$\text{where } \tau_{\alpha\beta}(x_1,x_2)\equiv\frac{1}{2h}\int_{-h}^{h}\tau_{\alpha\beta}(x_1,x_2,x_3)\,dx_3.\ (\alpha,\beta=1,2) \tag{1·18}$$

$$F_\alpha(x_1,x_2)\equiv\frac{1}{2h}\int_{-h}^{h}F_\alpha(x_1,x_2,x_3)\,dx_3.$$

Taking the mean values in the stress-strain relations (1.9) we get

$$\bar{\tau}_{\alpha\beta}=\bar{\lambda}\bar{\vartheta}\delta_{\alpha\beta}+\mu\left(\bar{u}_{\alpha,\beta}+\bar{u}_{\beta,\alpha}\right),\ (\alpha,\beta=1,2) \tag{1·19}$$

$$\text{with } \bar{\lambda}\equiv\frac{2\lambda\mu}{\lambda+2\mu} \text{ and } \bar{\vartheta}\equiv\bar{u}_{\alpha,\alpha} \tag{1·20}$$

These, together with two equations (1.17), determine the five unknown mean values $\bar{u}_\alpha(x_1,x_2)$ and $\bar{\tau}_{\alpha\beta}(x_1,x_2)$.

Putting (1.20) in (1.18) we get two equations of Navier type, viz.

$$\mu\nabla^2\bar{u}_\alpha+\left(\bar{\lambda}+\mu\right)\frac{\partial\bar{\vartheta}}{\partial x_\alpha}+\bar{F}_\alpha(x_1,x_2)=0,\ (\alpha=1,2) \tag{1·21}$$

from which the average displacements $\bar{u}_\alpha$ can be determined.

Again taking the average stresses the Beltrami-Michell compatibility equations turn out to be

$$\nabla^2 \bar{\Theta}_1 = -\frac{2(\bar{\lambda}+\mu)}{\bar{\lambda}+2\mu}.\bar{F}_{\alpha,\alpha}, \ (\alpha = 1,2) \tag{1·22}$$

where $\bar{\Theta}_1 = \bar{\tau}_{11} + \bar{\tau}_{22}$

The equation together with the equilibrium equations (1.17), determine the mean stresses $\bar{\tau}_{\alpha\beta}$ with the boundary conditions on the edges given by

$$\tau_{\alpha\beta}\nu_\beta = T_\alpha(p) \tag{1·23}$$

Integrating these with respect to x_3 between the limits $-h$ and $+h$ and then dividing by $2h$ gives

$$\bar{\tau}_{\alpha\beta}\nu_\beta = \bar{T}_\alpha(s), \ (\alpha,\beta = 1,2) \text{ on the boundary } C. \tag{1·24}$$

where $T_\alpha(s)ds$ are the components of the force applied on the arc ds of the contour C.

The two-dimensional boundary-value problem comprising the system of equations (1.17), (1.22) and (1.24) is known as **generalized plane stress problem**.

1.4 Plane Elastostatic Problem

It is obvious from discussions in sections 1.2 and 1.3 that the mathematical formulations of plane deformation and generalized plane stress problems are identical, the only difference is the appearance of barred symbols : $\bar{u}_\alpha, \bar{\tau}_{\alpha\beta}, \bar{\tau}, \bar{T}_\alpha$, etc. Therefore, we refer these two types as plane elastostatic problems.

The treatment of plane problems of elasticity is simplified to some extent if the body forces F_α does not appear in the differential equations. However in the case of appearance of the body forces, we may take

$$\tau_{\alpha\beta} = \tau^{(0)}_{\alpha\beta} + \tau^{(1)}_{\alpha\beta}, \quad (\alpha,\beta = 1,2)$$

where $\tau^{(0)}_{\alpha\beta}$ is any set of functions satisfying equilibrium equations

$$\tau^{(0)}_{\alpha\beta,\beta} + F_\alpha = 0 \ \ (\alpha,\beta = 1,2) \tag{1·25}$$

and $\tau^{(1)}_{\alpha\beta}$ satisfy the homogeneous equations

$$\tau^{(1)}_{\alpha\beta,\beta} = 0 \quad (\alpha,\beta = 1,2) \tag{1·26}$$

Thus we see that when there are body forces F_α, we find particular solution $\tau^{(0)}_{\alpha\beta}$ of the equations

$$\tau_{\alpha\beta,\beta} = -F_\alpha$$

and the solution $\tau^{(1)}_{\alpha\beta}$ of the homogeneous equations

$$\tau_{\alpha\beta,\beta} = 0$$

Similar considerations are also applied to the Nairer equations (1.12). Then the general solution will be

$$\tau_{\alpha\beta} = \tau^{(0)}_{\alpha\beta} + \tau^{(1)}_{\alpha\beta}.$$

For example, in the case of constant gravitational force F_α directed along the x_2-axis, we have $F_1 = 0$, $F_2 = -g\rho$ where ρ is the density unit volume and g is the gravitational acceleration.

The particular solutions $\tau^{(0)}_{\alpha\beta}$ of (1.24) are then

$$\tau^{(0)}_{11} = 0 = \tau^{(0)}_{12}$$

$$\tau^{(0)}_{22} = \rho g x_2$$

$$u^{(0)}_1 = -\frac{\lambda}{u\mu(\lambda+\mu)}\rho g\, x_1 x_2$$

$$u^{(0)}_2 = \frac{\lambda+2\mu}{8\mu(\lambda+\mu)}\rho g x_2^2 + \frac{\lambda}{8\mu(\lambda+\mu)}\rho g\, x_1^2$$

1.5 Airy's Stress Function

We have already noted that the boundary value problems in plane elasticity can always be reduced to those in which the body forces are absent. Accordingly, we consider the equlibrium equations in the form.

$$\tau_{\alpha\beta,\beta} = 0\,,\ (\alpha.\beta = 1.2) \tag{1·27}$$

in which $\tau_{\alpha\beta}$ satisfy the compatibility equation

$$\nabla_1^2(\tau_{11} + \tau_{22}) = 0 \tag{1·28}$$

in the region R and on the boundary C of R we have

$$\tau_{\alpha\beta}\nu_\beta = T_\alpha(s) \tag{1·29}$$

in which ν_β represent the components of the exterior unit normal vector to C and $T_\alpha(s)$ are known functions of the arc parameter s of C.

Now, if we introduce a function $U(x_1, x_2)$ such that

$$\tau_{22} = U_{,11}, \quad \tau_{12} = -U_{,12} \quad \text{and} \quad \tau_{11} = U_{,22} \tag{1·30}$$

then it can be easily seen that equilibrium equations (1.27) are satisfied identically. The compatibility equations (1.28) will be satisfied if

$$\nabla_1^2 \nabla_1^2 U = 0 \text{ in the region } R. \tag{1·31}$$

Equation (1.31) is a biharmonic equation involving the biharmonic function $U = U(x_1, x_2)$ and is known as **Airy's stress function.**

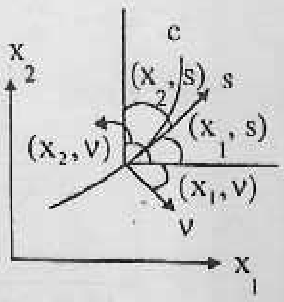
Now, in terms of the biharmonic function U, the boundary conditions (1.28) give.

$$\left.\begin{aligned} U_{,22}\nu_1 - U_{,12}\nu_2 &= T_1(s) \\ -U_{,12}\nu_1 + U_{,11}\nu_2 &= T_2(s) \end{aligned}\right\} \tag{1·31}$$

Noting that

$$\nu_1 = \cos(x_1, \nu) = \cos(x_2, s) = \frac{dx_2}{ds}$$

$$\nu_2 = \cos(x_2, \nu) = -\cos(x_1, s) = -\frac{dx_1}{ds}$$



we have from (1.31)

$$\frac{\partial}{\partial x_2}(U_{,2})\frac{dx_2}{ds} + \frac{\partial}{\partial x_1}(U_{,2})\frac{dx_1}{ds} = T_1(s)$$

$$-\left[\frac{\partial}{\partial x_2}(U_{,1})\frac{dx_2}{ds} + \frac{\partial}{\partial x_1}(U_{,1})\frac{dx_1}{ds}\right] = T_2(s)$$

$$\text{i.e.} \frac{d}{ds}(U_{,2}) = T_1(s) \tag{1·33}$$

$$\text{and } -\frac{d}{ds}(U_{,1}) = T_2(s)$$

Integrating the above equations with respect to s along C from some fixed point s_0 on C to a variable point s, we get

$$U_{,1}(s) = -\int_{s_0}^{s} T_2(s)ds \equiv f_1(s) + C_1 \tag{1·34}$$

$$U_{,2}(s) = -\int_{s_0}^{s} T_1(s)ds \equiv f_2(s) + C_2$$

It is clear from (1.34) that the derivatives of U along C are not determined uniquely because of the arbitrary constants C_1 and C_2. However, stresses are determined uniquely as they involve second derivatives in U. Thus we see that the two dimensional boundary value problem governed by equations (1.27), (1.28) and (1.29) is related to the boundary value problem of the type :

$$\nabla_1^4 U = 0 \quad \text{in } R \tag{1·35}$$

$$U_{,\alpha} = f_\alpha(s) \text{ on } C, \ (\alpha = 1,2)$$

where $f_\alpha(s)$ are certain known functions.

The problem (1.35) may also be put in a slightly different way as follows :

Since

$$\frac{dU}{d\nu} = U_{,\alpha}\nu_\alpha = U_{,1}(s)\frac{dx_2}{ds} - U_{,2}(s)\frac{dx_1}{ds}$$

$$\equiv g(s), \text{ a known function on } C,$$

and, also

$$dU = U_{,\alpha}dx_\alpha = f_\alpha dx_\alpha = f_\alpha \frac{dx_\alpha}{ds}ds$$

$$\Rightarrow U = \int_{s_0}^{s} f_\alpha \frac{dx_\alpha}{ds}ds \equiv f(s) + \text{constant}.$$

Thus (1.35) be put in a different, but equivalent form as :

$$\nabla_1^4 U = 0 \text{ in R}$$

$$U = f(s) + \text{constant on } C.$$

$$\frac{du}{d\nu} = g(s) \text{ on } C$$

which is more convenient in some investigations.

General solution of biharmonic equation :

Let us consider the biharmonic equation

$$\nabla_1^4 U = \nabla_1^2\nabla_1^2 U = 0 \text{ in } R \tag{1.36}$$

and suppose that $\nabla_1^2 U = P_1(x_1, x_2)$

then $\nabla_1^2 P_1 = 0$, and so the function P_1 is harmonic in R.

We construct a function $F(z) = P_1 + iP_2$ of a complex variable $z\left(= x_1 + ix_2\right)$ by computing from P_1, its conjugate $P_2\left(x_1, x_2\right)$ as following :

We have $dP_2 = P_{2,1}dx_1 + P_{2,2}dx_2$.

By Cauchy-Riemann equations,

$$P_{2,1} = -P_{1,2}$$

$$P_{2,2} = P_{1,1}$$

so that $dP_2 = -P_{1,2}dx_1 + P_{1,1}dx_2$

Hence $P_2 = \int\left(-P_{1,2}\,dx_1 + P_{1,1}dx_2\right)$

We define a function $\phi(z)$ by

$$\phi(z) = \tfrac{1}{4}\int F(z)\,dz \qquad (1.37)$$

$$= p_1 + ip_2$$

Clearly, the function $\phi(z)$ is analytic and, therefore,

$$\phi'(z) = \frac{\partial p_1}{\partial x_1} + i\frac{\partial p_2}{\partial x_1} = \tfrac{1}{4}\left(P_1 + iP_2\right)$$

Equating real and imaginary parts we get

$$\frac{\partial p_1}{\partial x_1} = \tfrac{1}{4}P_1,\ \frac{\partial p_2}{\partial x_2} = \tfrac{1}{4}P_2$$

Also from Canchy-Riemann equations

$$p_{1,1} = p_{2,2},\ p_{1,2} = -p_{2,1}$$

so that $p_{1,1} = p_{2,2} = \tfrac{1}{4}P_1$

and $p_{1,2} = -p_{2,1} = -\tfrac{1}{4}P_2$

using these results and noting that p_1 and p_2 are harmonic functions in R, we may verify that

$$\nabla_1^2\left(U - p_1x_1 - p_2x_2\right) = 0 \text{ in } R$$

so that, $U = p_1x_1 + p_2x_2 + q_1\left(x_1, x_2\right)$ (1.38) where $q_1(x_1, x_2)$ is harmonic in R.

As before we construct another analytic function $\chi(z) = q_1 + iq_2$, whose real part is q_1.

Now,

$$\bar{z}\phi(z) + \chi(z) = (x_1 - ix_2)(p_1 + ip_2) + q_1 + iq_2$$

$$= (p_1x_1 + p_2x_2 + q_1) + i(p_2x_1 - p_1x_2 + q_2) \quad (1.39)$$

Hence from (1.38) and (1.39) we see that

$$U = \mathrm{Re}\left[\bar{z}\phi(z) + \chi(z)\right] \quad (1.40)$$

where Re donotes the real part of the function.

Again we have

$$\overline{\phi(z)} = p_1 - ip_2$$

$$\overline{\chi(z)} = q_1 - iq_2$$

and $z\overline{\phi(z)} + \overline{\chi(z)} = (x_1 + ix_2)(p_1 - ip_2) + q_1 - iq_2$

$$= (p_1x_1 + p_2x_2 + q_1) + i(p_1x_2 - p_2x_1 - q_2)$$

Adding (1.39) and (1.41), we get (1.41)

$\bar{z}\phi(z) + z\overline{\phi(z)} + \chi(z) + \overline{\chi(z)} = 2U$ by (1.38)

Hence,

$$2U = \bar{z}\phi(z) + z\overline{\phi(z)} + \chi(z) + \overline{\chi(z)} \quad (1.42)$$

Hence the solution of the biharmonic equation is given by (1.38) or by (1.42). In (1.42) it is obtained in terms of two analytic harmonic functions $\phi(z)$ and $\chi(z)$. The formula (1.42) is known as **Goursat formula.**

Formulae for displacements and stresses :

From (1.29) we have the stress components

$$\tau_{11} = U_{,22},\ \tau_{22} = U_{,11} \text{ and } \tau_{12} = U_{,12} \quad (1.43)$$

Also from Coursat formula

$$2U = \bar{z}\phi(z) + z\overline{\phi(z)} + \chi(z) + \overline{\chi(z)} \quad (1.44)$$

Now,

$$\tau_{11} + i\tau_{12} = U_{,22} - iU_{,12}$$

$$= -i(U_{,1} + iU_{,2})_{,2} \tag{1.45}$$

and $\tau_{22} - i\tau_{12} = U_{,11} + iU_{,12} = (U_{,1} + iU_{,2})_{,1}$ (1.46)

From (1.44), it follows that

$$U_{,1} + iU_{,2} = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{1.47}$$

where $\psi(z) = \chi'(z)$ (1.48)

Then we get from (1.46) and (1.47)

$$\tau_{11} + i\tau_{12} = \phi'(z) + \overline{\phi'(z)} - z\overline{\phi''(z)} - \overline{\psi'(z)}$$

$$\tau_{11} + i\tau_{12} = \phi'(z) + \overline{\phi'(z)} + z\overline{\phi''(z)} + \overline{\psi'(z)}$$

From these we have

$$\tau_{11} + \tau_{22} = 2\left[\phi'(z) + \phi'(z)\right] = 4\,\mathrm{Re}\left[\phi'(z)\right]$$

and

$$\tau_{22} - \tau_{11} + 2i\tau_{12} = U_{,11} + U_{,22} - 2iU_{,12} = \frac{\partial^2 U}{\partial z^2}$$

$$= 2\left[\overline{z}\phi''(z) + \psi'(z)\right] \tag{1.49}$$

The first equation of (1.49) gives $\tau_{11} + \tau_{22}$ and second equation, on equating real and imaginary parts, gives $\tau_{22} - \tau_{11}$ and τ_{12} and hence τ_{11}, τ_{22} and τ_{12} are all known in terms of two harmonic functions $\phi(z)$ and $\chi(z)$. The formula (1.49) is known as **Kalosou-Muskhilisivili formulae.**

Let us now proceed to derive formula for the displacement components u_1 and u_2 in terms of $\phi(z)$ and $\psi(z)$. We have

$$\tau_{11} = U_{,22} = \lambda\vartheta + 2\mu u_{1,1} \tag{1.50}$$

$$\tau_{22} = U_{,11} = \lambda\vartheta + 2\mu u_{2,2}$$

$$\tau_{12} = U_{,12} = \mu(u_{1,2} + u_{2,1})$$

Since $\vartheta = u_{1,1} + u_{2,2}$, so from he first two equations we get

$$\tau_{11} + \tau_{22} = 2(\lambda + \mu)\vartheta,$$

i.e. $$\vartheta = \frac{\tau_{11} + \tau_{22}}{2(\lambda + \mu)} = \frac{\nabla_1^2 U}{2(\lambda + \mu)}$$

and hence

$$2\mu u_{1,1} = -U_{,11} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} . \nabla_1^2 U,$$

$$2\mu u_{2,2} = -U_{,22} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} . \nabla_1^2 U.$$

Also $\nabla^2 U = P_1 = 4p_{1,1} = 4p_{2,2}$

Thus,

$$2\mu\ u_{1,1} = -U_{,11} + \frac{2(\lambda + 2\mu)}{(\lambda + \mu)} p_{1,1}$$

$$2\mu\ u_{2,2} = -U_{,22} + \frac{2(\lambda + 2\mu)}{(\lambda + \mu)} p_{2,2}$$

whose integrations lead to

$$2\mu\ u_1 = -U_{,1} + \frac{2(\lambda + 2\mu)}{(\lambda + \mu)} p_1 + f(x_2) \tag{1.51}$$

$$2\mu\ u_2 = -U_{,2} + \frac{2(\lambda + 2\mu)}{(\lambda + \mu)} p_2 + g(x_1)$$

where $f(x_2)$ and $g(x_1)$ are arbitrary functions of x_2 and x_1 respectively.

Then,

$$2\mu(u_{1,2} + u_{2,1}) = -U_{,12} - U_{,21} + \frac{2(\lambda + 2\mu)}{(\lambda + \mu)}(p_{1,2} + p_{2,1})$$

$$+ f'(x_2) + g'(x_1).$$

or, $\mu(u_{1,2} + u_{2,1}) = -U_{,12} + \frac{1}{2}(f'(x_2) + g'(x_1))$(since $p_{1,2} = -p_{2,1}$)

So from 3rd equation of (1.50) we find that

$$f'(x_2) + g'(x_1) = 0$$

or, $f'(x_2) = -g'(x_1) = \text{constant} = \alpha(\text{say})$

$$\therefore f(x_2) = \alpha x_2 + \beta$$

$$g(x_1) = -\alpha x_1 + \gamma$$

where β and γ are constants.

The forms of f and g indicate that they represent rigid body displacements and so far elastic deformation we must put $f = g = 0$. Now from (1.51) we get

$$2\mu(u_1 + iu_2) = -(U_{,1} + iU_{,2}) + \frac{2(\lambda+2\mu)}{\lambda+\mu}(p_1 + ip_2)$$

$$= -\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right] + \frac{2(\lambda+2\mu)}{\lambda+\mu}\phi(z)$$

$$= \frac{\lambda+3\mu}{\lambda+\mu}\phi(z) - z\overline{\phi'(z)} - \overline{\psi'(z)}$$

$$= (3-4\sigma)\phi(z) - z\overline{\phi'(z)} - \overline{\psi'(z)} \qquad (1.52)$$

where $\frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\sigma$, $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ is the Poisson's ratio of the material.

Thus, the displacement components u_1 and u_2 are expressed in terms of two harmonic functions $\phi(z)$ and $\psi(z)$ in equation (1.52).

1.6 Summary

The two-dimesional boundary value problems have been considered in this unit. Plane strain, plane stress and generalized plane stress are discussed. The solution of the boundary value problem in the absence of body forces has been represented in terms of Airy's stress function. Filally, the solution of the analogous problem is obtained in terms of analytic functions.

1.7 Exercises

1. Short Questions :

(a) When will you call an elastic body to be in the state of plane strain/plane stress/generalized plane stress?

(b) Write down the expressions for stresses in terms of strain in the case of plane stress problem.

(c) Derive the stress-strain relations given in terms of their mean values.

(d) Obtain the displacement components u_1 and u_2 in terms of two harmonic functions $\phi(z)$ and $\psi(z)$.

2. ***Broad answer type :***

(a) Explain how a plane problem of elasicity can be solved by using Airy's stress function.

(b) Obtain Kolosou's formulae for stresses and displacements of plane problems of elasticity.

(c) Obtain a general solution of biharmonic equation in terms of analytic functions.

(d) If a uniform centrifugal force acts on a body, rotating with constant angular velocity about x_3-axis then find stresses and displacement for particular integral.

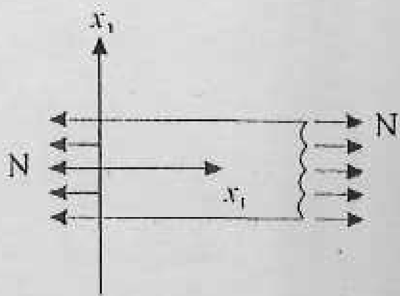
(e) Derive the Goursat formula.

Unit 2 □ Extension and Torsion

Structure

2.1 Axial Extension of a Beam

Consider an elastic beam of uniform cross-section bounded by a cylindrical surface and by a pair of planes normal to this surface. The cylindrical surface is called the lateral surface of the beam and the planes are the bases (or end faces). Also suppose that the beam is in equilibrium under the action of a uniform normal stress N acting on the bases and no body forces. Our problem is to find the stresses, strains and displacements at an arbitrary point in the beam.



Let us choose the x_1-axis along the line of centroids of the cross-sections of the beam of initial length l and take the origin on one of the bases. The bases of the beam are given by $x_1=0$ and $x_1=l$ before deformation.

The bounding conditions of the problem are

$$\tau_{11} = N,\ \tau_{12} = \tau_{13} = 0 \text{ for } x_1 = 0, l \tag{2.1}$$

and $\tau_{ij}\nu_j = 0$ on the lateral surface.

We note that on the lateral surface $\nu_1 = 0$ (since the normal to the lateral surface is perpendicular to the x_1-axis). Hence if we take

$$\tau_{11} = N_1 \ \ \tau_{12} = \tau_{23} = \tau_{31} = \tau_{22} = \tau_{33} = 0, \tag{2.2}$$

then we see that the boundary conditions (2.1) for the problem are satisfied.

The stress system (2.2) satisfies the equations of equilibrium $\tau_{ij,j} = 0$ (since N is constant here).

Also the Beltrami-Michell compatibility equations

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x_i \partial x_j} = 0 \quad (i, j = 1, 2, 3),$$

σ being the Poisson's ratio and $\Theta = \tau_{ii}\,(i = 1,2,3)$, are satisfied by the stress system (2.2).

Hence we conclude that the stress system given in (2.2) represents the stresses at an arbitrary point in the beam.

The strain components can be easily oblained from the stress strain relations as

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\,\Theta, \ (i, j = 1, 2, 3),$$

$$e_{11} = \frac{1}{E}\left[\tau_{11} - \sigma\left(\tau_{22} + \tau_{33}\right)\right] = \frac{N}{E}$$

$$e_{22} = \frac{1}{E}\left[\tau_{22} - \sigma\left(\tau_{11} + \tau_{33}\right)\right] = \frac{-\sigma N}{E} = e_{33} \tag{2.3}$$

$$e_{12} = \frac{1+\sigma}{E}\tau_{12} = 0, \ e_{23} = 0, \ e_{31} = 0$$

where E is the Young's modulus.

Now, we have by using the relation $e_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$,

$$e_{11} = \frac{N}{E} = \frac{\partial u_1}{\partial x_1} \tag{2.4}$$

$$e_{22} = \frac{-\sigma N}{E} = \frac{\partial u_2}{\partial x_2}$$

$$e_{33} = \frac{-\sigma N}{E} = \frac{\partial u_3}{\partial x_3}.$$

$$\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}=0,\ \frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}=0,\ \frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}=0$$

Solving first three equations of (2.4) we get

$$u_1=\frac{Nx_1}{E}+a_1x_2+b_1x_3+c_1$$

$$u_2=\frac{-\sigma}{E}Nx_2+a_2x_3+b_2x_1+c_2 \qquad (2.5)$$

$$u_3=\frac{-\sigma}{E}Nx_3+a_3x_1+b_3x_2+c_3$$

Using the last three conditions of (2.4) we get

$$a_1=-b_2,\ a_2=-b_3,\ a_3=-b_1 \qquad (2.6)$$

Since there is no rigid body rotation we must have

$$u_{i,j}=u_{j,i} \qquad (2.7)$$

These give

$$a_1=b_2,\ a_2=b_3,\ a_3=b_1 \qquad (2.8)$$

(2.6) and (2.8) implies that

$$a_1=a_2=a_3=b_1=b_2=b_3=0 \qquad (2.9)$$

Since there is no displacement at the origin i.e.

$$u_1=u_2=u_3=0 \text{ at } (0,0,0)$$

and so $c_1=c_2=c_3=0$ (2.10)

Therefore, the displacement components are

$$u_1=\frac{Nx_1}{E}$$

$$u_2=\frac{-\sigma}{E}Nx_2 \qquad (2.11)$$

$$u_3=\frac{-\sigma}{E}Nx_3.$$

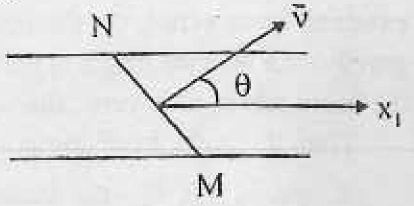
Example : In the problem we have considered above find the normal and shearing streses on oblique section of the beam. Also find their extreme values.

Solution : Let MN be an oblique section of the beam whose normal makes an angle θ with x_1-axis.

Then $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$ where $\nu_1 = \cos\theta$

Using the relation $\overset{\nu}{T}_i = \tau_{ij}\nu_j$, we get

$$\overset{\nu}{T}_1 = \tau_{1j}\nu_j = \tau_{11}\nu_1 + \tau_{12}\nu_2 + \tau_{13}\nu_3 = N\nu_1 = N\cos\theta$$



$$\overset{\nu}{T}_2 = \tau_{2j}\nu_j = \tau_{21}\nu_1 + \tau_{22}\nu_2 + \tau_{23}\nu_3 = 0$$

$$\overset{\nu}{T}_3 = 0.$$

If $\overset{\nu}{T}_n$ represents the normal stress on the plane whose normal is $\vec{\upsilon}$ then

$$\overset{\nu}{T}_n = \overset{\nu}{T}_1\nu_1 + \overset{\nu}{T}_2\nu_2 + \overset{\nu}{T}_3\nu_3 = N\cos\theta.\cos\theta = N\cos^2\theta$$

The resultant traction on the oblique plance is

$$T = \sqrt{\left(\overset{\nu}{T}_1\right)^2 + \left(\overset{\nu}{T}_2\right)^2 + \left(\overset{\nu}{T}_3\right)^2} = N\cos\theta.$$

Hence the shearing stress $\overset{\nu}{T}_s$ on the plane is obtained from the relation

$$\left(\overset{\nu}{T}_n\right)^2 + \left(\overset{\nu}{T}_s\right)^2 = T^2$$

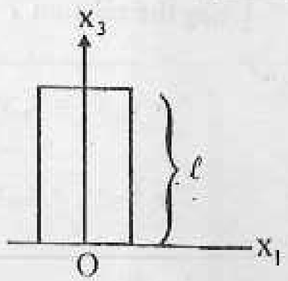
as $\overset{\nu}{T}_s = \left((N\cos\theta)^2 - \left(N\cos^2\theta\right)^2\right)^{1/2} = \left(N^2\cos^2\theta\ \sin^2\theta\right)^{1/2}$

$$\therefore \overset{\nu}{T}_s = \tfrac{1}{2}N\,|\sin 2\theta|.$$

Since $\overset{\nu}{T}_n = N\cos^2\theta = \frac{1}{2}N(1+\cos 2\theta)$, so

$\overset{\nu}{T}_n$ will be maximum when 0 and it will be minimum when $\theta = 90°$. Also $\overset{\nu}{T}_s$ will be maximum when $\theta = 45°$ and it will be minimum when $\theta = 0$.

2.2 Beam stretched by its Own Weight



Let us consider a beam of length l in its undeformed state. Suppose that the beam is supported at its upper base as shown in the figure and the only external force acting on the beam is the force of gravity. We take the origin at the lower base x_3=0 of the beam and x_3-axis vertically upwards.

Then the body force component are

$F_1 = 0$, $F_2 = 0$, $F_3 = \rho g$ where ρ is the density of the beam.

The stress system

$$\tau_{33} = \rho g x_3,\ \tau_{11} = \tau_{22} = \tau_{12} = \tau_{23} = \tau_{31} = 0 \tag{2.12}$$

satisfies the stress equations of equilibrium

$$\tau_{ij,j} + F_i = 0 \quad (i, j = 1,2,3)$$

and compatibility equations

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = -\frac{\sigma}{1+\sigma}\delta_{ij} div\vec{F} - \left(F_{i,j} + F_{j,i}\right).$$

Also the system (2.12) satisfies the boundary condition that the lateral surface (where normal is perpendicular to x_3-axis) is free from traction. The lower base is also free from traction, since at the lower base $x_3 = 0$, $\tau_{33} = 0$. But at the upper base $\tau_{33} = \rho g\ell$, which is directed vertically upward. Thus the cylinder is to be supported in such a way as to give a uniform distribution of stress. To find the displacement components we have

$$\frac{\partial u_1}{\partial x_1} = e_{11} = \frac{1}{E}\left[\tau_{11} - \sigma\left(\tau_{22} + \tau_{33}\right)\right] = -\frac{\sigma}{E}\rho g x_3 \quad (a)$$

$$\frac{\partial u_2}{\partial x_2} = e_{22} = \frac{1}{E}\left[\tau_{22} - \sigma\left(\tau_{11} + \tau_{33}\right)\right] = \frac{-\sigma}{E}\rho g x_3 \quad (b) \tag{2.13}$$

$$\frac{\partial u_3}{\partial x_3} = e_{33} = \frac{1}{E}\left[\tau_{33} - \sigma\left(\tau_{11} + \tau_{22}\right)\right] = \frac{\rho g x_3}{E} \quad (c)$$

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0,\ \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0,\ \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0 \tag{2.14}$$

Integrating [2.13(c)] we get

$$u_3 = \frac{\rho g x_3^2}{2} + u_{30}(x_1, x_2)$$

where u_{30} is a function of x_1 and x_2 only and from last two equations of (2.14) it follows that

$$u_1 = -x_3 \frac{\partial w_0}{\partial x_1} + u_{10}(x_1, x_2)$$

$$u_2 = -x_3 \frac{\partial w_0}{\partial x_3} + u_{20}(x_1, x_2)$$

where u_{10}, u_{20} are functions of x_1 and y_1 only.

Substituting the values of u_1 and u_2 in [2.13(9)] and [2.13(b)] we get

$$\frac{\partial u_{10}}{\partial x_1} = 0, \ \frac{\partial u_{20}}{\partial x_2} = 0 \qquad (2.15)$$

$$\frac{\partial^2 u_{30}}{\partial x_1^2} = \frac{\sigma \rho g}{E}, \ \frac{\partial^2 u_{30}}{\partial x_2^2} = \frac{\sigma \rho g}{E} \qquad (2.16)$$

Also substituting u_1 and u_2 into the first equation of (2.14) we get

$$\frac{\partial^2 u_{30}}{\partial x_1 \partial x_2} = 0, \ \frac{\partial u_{10}}{\partial x_2} + \frac{\partial u_{20}}{\partial x_1} = 0 \qquad (2.17)$$

we see from (2.15) that $u_{10} = F(x_2)$, a function of x_2 alone and $u_{20} = G(x_1)$, a function of x_1 alone.

From (2.17), we get

$F'(x_2) = -G'(x_1) = \text{constant} = a$ (say)

$\therefore u_{10} = ax_2 + b$

$u_{20} = -ax_1 + c$

From (2.16) and (2.17) we easily find that

$$u_{30} = \frac{\sigma \rho g}{2E}\left(x_1^2 + x_2^2\right) + a' x_1 + b' x_2 + c'$$

where a', b', c' are constants.

Finally, the displacement components are

$$u_1 = -\frac{\sigma \rho g}{E} x_3 x_1 - a' x_3 + ax_2 + b$$

$$u_2 = -\frac{\sigma \rho g}{E} x_3 x_2 - b' x_3 - ax_1 + c \qquad (2.18)$$

$$u_3=\frac{\rho g}{2E}\left(x_3^2+\sigma x_1^2+\sigma x_2^2\right)+a'x_1+b'x_2+c'$$

To determine the constants in (2.18) we may assume that there is no displacement at the point $(0,0,\ell)$, ie $u_1=u_2=u_3=0$ at $(0,0,\ell)$. This gives

$$b=a'\ell,\ c=b'\ell,\ c'=-\frac{\rho g\ell^2}{2E}. \tag{2.19}$$

Also we can avoid rigid body rotation, by noting that the rotation components are zero at $(0,0,\ell)$

i.e. $\frac{\partial u_2}{\partial x_3}=\frac{\partial u_3}{\partial x_2},\ \frac{\partial u_3}{\partial x_1}=\frac{\partial u_1}{\partial x_3},\ \frac{\partial u_1}{\partial x_2}=\frac{\partial u_2}{\partial x_1}$ at $(0,0,\ell)$

Applying these condtions we get

$b'=0,\ a'=0\ and\ a=0$

so from (2.19) $b=c=0$

Hence the displacement components associated with the problem are given by

$$u_1=-\frac{\sigma\rho g}{E}x_3x_1, \tag{2.20}$$

$$u_2=-\frac{\sigma\rho g}{E}x_3x_2$$

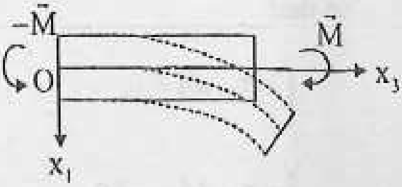
$$u_3=\frac{\rho g}{2E}\left(x_3^2+\sigma x_1^2+\sigma x_2^2-\ell^2\right)$$

2.3 Bending of a Beam by Terminal Couples

Let us consider an elastic beam of uniform cross-section and of length l. Suppose that a couple of moment $\vec{M}$ about a line perpendicular to the beam is applied at one of the bases and an opposite couple of moment $-\vec{M}$ is applied at the other base so that the beam is in equilibrium. The lateral surface of the beam is stress free and the body forces are neglected. The problem is to find the stresses, strains and displacements that occur at on arbitrary point of the beam due to the bending it experiences because of the applied end couples. This kind of bending is known as **pure bending**.

We take the centroid of the base of the beam on which the couple $-\vec{M}$ acts as the origin of the co-ordinate system with x_3-axis initially directed along the line of the centroids of cross-section of the beam. Also the x_2-axis is chosen along the axis of the couple $\bar{M}$ and the x_1-axis is chosen such that the co-ordinate system is right handed.

Due to the bending caused by the applied couples, the longitudinal elements of the beam experiences elongation or contraction. Consequently, the stress vector $\overset{\nu}{\vec{T}}$ acting at a point on a cross-section of the beam produces a moment $\vec{x}\times\overset{\nu}{\vec{T}}$ per unit of area.



The total moment on the cross section is $\int_A \left(\vec{x}\times\overset{\nu}{\vec{T}}\right) dA$, where A is the area of the cross-section.

On the cross-section $x_3 = \ell$, we have

$$\int_A \left(\vec{x}\times\overset{\nu}{\vec{T}}\right) dA = \bar{M} \tag{2.21}$$

We note that on the section $x_3 = \ell$, $\vec{x} = (x_1, x_2, \ell)$, $\bar{\nu} = (0,0,1)$, $\overset{\nu}{\vec{T}} = (\tau_{31}, \tau_{32}, \tau_{33})$ and $\bar{M} = (0, M, 0)$ where $M = |\bar{M}|$.

Hence from (2.21)

$$\int_A (x_1, x_2, \ell)\times(\tau_{31}, \tau_{32}, \tau_{33})\, dA = (0, M, 0)$$

This gives, on $x_3 = \ell$.

$$\int_A (x_2\tau_{33} - \ell\tau_{32})\, dA = 0,$$

$$\int_A (\ell\tau_{31} - x_1\tau_{33})\, dA = M, \tag{2.22}$$

$$\int_A (x_1\tau_{32} - x_2\tau_{31})\, dA = 0.$$

Similarly, on $x_3 = 0$, we note that at a point with co-ordinates $(x_1, x_2, 0)$, $\vec{v} = (0,0,-1)$, $\bar{M} = (0,-M,0)$,

so that

$$\int_A x_2 \tau_{33}\, dA = 0,$$

$$\int_A x_1 \tau_{33}\, dA = -M, \tag{2.23}$$

$$\int_A (x_1 \tau_{32} - x_2 \tau_{31})\, dA = 0.$$

The conditions (2.22) and (2.23) are to be satisfied by the stresses at $x_3 = \ell$ and $x_3 = 0$ respectively.

We may verify that the stress system

$$\tau_{11} = \tau_{22} = \tau_{12} = \tau_{13} = \tau_{23} = 0,\ \tau_{33} = ax_1 \tag{2.24}$$

where a' is a non-zero constant satisfies all the conditions (2.22) and (2.23), provided that

$$\int_A x_1 x_2\, dA = 0 \tag{2.15}$$

and $\int_A x_1^2 dA = -\frac{M}{a}$ (2.26)

Since $I = \int_A x_1^2 dA$ is the moment of inertia of a section about x_2-axis, so we have from (2.26) $a = -\frac{M}{I}$.

Equation (2.25) requires that the product of inertia of a section with respect ot x_1, x_2 axes is zero, so that the axes of x_1 and x_2 are the principal axes of inertia for the section. Thus the stress system (2.24) with $a = -\frac{M'}{I}$ satisfies all the conditions (2.22) and (2.23).

Again, on the lateral surface, we have $v_3 = 0$ and hence $\overset{v}{\vec{T}} = \vec{0}$ there. So the condition that the lateral surface is free from traction is satisfies by the stress system (2.24).

Finally, it is easy to verify that the stress system (2.24) satisfies the equilibrium equations and Beltrami-Michell compatibility conditions. Hence the stress system in the body due to pure bending is given by

$$\tau_{11} = \tau_{22} = \tau_{12} = \tau_{23} = \tau_{31} = 0$$

$$\tau_{33} = -\frac{M}{I}x_1 \qquad (2.27)$$

The strain components are obtained from the stress strain relations

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\theta$$

$$\text{as } e_{11} = e_{22} = \frac{\sigma M}{EI}x_1, \; e_{33} = -\frac{M}{EI}x_1 \qquad (2.28)$$

$$e_{12} = e_{23} = e_{31} = 0 .$$

It can be easily seen that the corresponding displacements are given, by using the relations $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, as

$$u_1 = \frac{M}{2EI}\left[x_3^2 + \sigma\left(x_1^2 - x_2^2\right)\right],$$

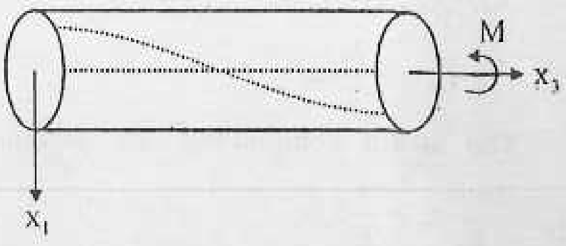
$$u_2 = \frac{M\sigma}{EI}x_1x_2 \qquad (2.29)$$

$$u_3 = \frac{-M}{EI}x_1x_2 .$$

We note from (2.28) that longitudinal elements of the beam extend or contract depending on whether $x_1 < 0$ or $x_1 > 0$. On $x_1 = 0$, all the strains are zero : as such the elements initially lying on the x_2x_3 plane do not change in length. The x_2x_3 plane is, therefore, referred to as the natural plance of the beam. Also $u_2 = 0$ on $x_2 = 0$ plane. The plane $x_2 = 0$ is referred to as the **plane of bending.** Again, we observe that the line element initially lying along the x_3-axis lies on the neutral plane as well as on the plane of bending. The x_3-axis is called the **Central line of the beam.**

2.4 Torsion of Cylindrical Bars of Circular Cross-section

Consider a circular cylinder of length l with one of its bases fixed in the x_1x_2-plane, while the other base $x_3=l$ is acted upon by a couple whose moment lies along the x_3-axis. Under the action of the couple, the beam will be twisted, and the generators of the cylinder will be deformed into helical curves. On account of symmetry of the cross-section, it is reasonable to suppose that the section of the cylindrical planes normal to the x_3-axis will remain plane after deformation and that the action of the couple will merely rotate each section through some angle θ, say. The amount of rotation will depend on the distance of the section from the base $x_3=0$ and since the deformation is small, it is assumed that the amount of rotation θ is proportional to the distance of the section from the fixed base. Thus



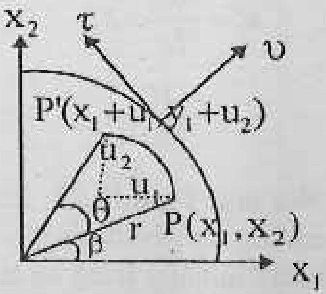
$$\theta = \alpha x_3,$$

where α is the twist per unit length, i.e., the relative angular displacement of a pair of cross-section that are unit distance apart.

If the cross-section of the cylinder remain plane after deformation then the displacement u_3, along x_3-axis, is zero. The displacements u_1 and u_2 are readily calculated as follows. Consider any point $P(x_1, x_2)$ in the circular section, which, before deformation, occuped the position shown in figure. After deformation, the point P will occupy a new position $P'(x_1+u_1, x_2+u_2)$. In terms of angular displacement θ of the point P, we have

$$u_1 = r\cos(\beta+\theta) - r\cos\theta$$
$$= x_1(\cos\theta - 1) - x_2\sin\theta$$

$$u_2 = r\sin(\beta+\theta) - r\sin\beta$$
$$= x_1\sin\theta + x_2(\cos\theta - 1)$$



Where β is the angle between the radius vector r and the x_1-axis so that $x_1 = r\cos\beta,\ x_2 = r\sin\beta$. If θ is small, we can write

$$u_1 = -\theta x_2,\ u_2 = \theta x_1$$

since $\theta = \tau x_3$, we have the displacement components at any point (x_1, x_2, x_3) in the cylinder as

$$u_1 = -\alpha x_3 x_2,\ u_2 = \alpha x_3 x_1,\ u_3 = 0. \qquad (2.30)$$

The system of stresses associated with the displacements (2.30) is given by

$$\tau_{32} = \mu\alpha x_1,\ \tau_{31} = -\mu\alpha x_2,\ \tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = 0 \qquad (2.31)$$

which obviously satisfy the equations of equilibrium under no body ferces and equations of compatibility. The boundary conditions on the lateral surface are satisfies the conditions

$$\tau_{11}\nu_1 + \tau_{12}\nu_2 = 0,$$
$$\tau_{21}\nu_1 + \tau_{22}\nu_2 = 0,$$
$$\tau_{31}\nu_1 + \tau_{32}\nu_2 = -\mu\alpha x_2 \cos(x_1, \nu) + \mu\alpha x_1 \cos(x_2, \nu)$$
$$= 0,$$

since, for a circle of radius a, $\cos(x_1, \nu) = \frac{x_1}{a}$ and $\cos(x_2, \nu) = \frac{x_2}{a}$.

The only non vanishing component of couple M produced by the distribution of stresses (2.31) over the end of cylinder is M_3, given by

$$M_3 = \iint (x_1\tau_{32} - x_2\tau_{13})\,dx_1 dx_2 = \mu\alpha \iint (x_1^2 + x_2^2)\,dx_1 dx_2$$
$$= \mu\alpha I_0$$

where $I_0 = \frac{\pi a^4}{2}$ is the polar moment of inertia of the circular section of radius a.

The resultant force acting on the end of the cylinder vanishes, and it follows from Saint Venant's principle that whatever be the distribution of forces, over the end of the cylinder that give rise to the couple of magnitude M_3, the distribution of stress sufficiently for from the ends of the cylinder is essentially specified by (2.31)

The stress vector

$$\bar{T} = \bar{i}\,\tau_{31} + \bar{j}\,\tau_{32} + \bar{k}\,\tau_{33} = \mu\alpha(-\bar{i}x_2 + \bar{j}x_1)$$

acting at a point (x_1, x_2) on any cross-section x_3-constant lies in the plane of section and is normal to the radius vector r joining the point (x_1, x_2) with the origin (0,0). The magnitude of $\bar{T}$ is

$$\tau = \sqrt{\tau_{31}^2 + \tau_{32}^2} = \mu\alpha\sqrt{x_1^2 + x_2^2} = \mu\alpha r$$

The maximum stress is a tangential stress that acting on the boundary of the cylinder and has the magnitude $\mu\alpha a$, i.e.,

$$\tau_{max} = \mu\alpha a.$$

2.5 Torsion of a Cylindrical Bar of any Given Section

Consider a cylindrical bar of any cross-section subjected to no body forces and free from lateral traction. One end of the bar is fixed in the plance x_3=0, while the other end i.e. the plane $x_3 = \ell$ is twisted by a couple of magnitude M whose moment is directed along the axis of the bar. Clearly x_3-axis is taken along the axis of the bar.

By the moment condition on the end $x_3 = \ell$,

$$M = M_3 = \int_s (x_1\tau_{23} - x_2\tau_{31})\,ds \text{ on } x_3 = \ell \qquad (2.32)$$

Following Saint-Venant semi-inverse method the stress components that do not contribute to the moment M in (2.32) may be assumed to be zero throughout the body, i.e.

$$\tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = 0 \qquad (2.33)$$

In view of stress-strain relations $e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\theta$,

$$e_{11} = e_{22} = e_{33} = e_{12} = 0 \qquad (2.34)$$

The stress equations of equilibrium are

$$\frac{\partial\tau_{13}}{\partial x_3} = 0,\ \frac{\partial\tau_{23}}{\partial x_3} = 0,\ \frac{\partial\tau_{13}}{\partial x_1} + \frac{\partial\tau_{23}}{\partial x_2} = 0 \qquad (2.35)$$

The first two equations of (2.35) implies that τ_{13} and τ_{23} are independent of x_3. The last equation of (2.35) can be replaced by writting

$$\tau_{13} = \mu\alpha\frac{\partial\Phi(x_1,x_2)}{\partial x_2},\ \tau_{23} = -\mu\alpha\frac{\partial\Phi(x_1,x_2)}{\partial x_1} \qquad (2.36)$$

where the stress function $\Phi(x_1,x_2)$ is a function of x_1 and x_2 and α is an adjustable non-zero constant taken for future convenience.

Stress-compatibility conditions under no body forces are

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\ \Theta_{,ij} = 0\ (i\,,j = 1,2,3) \qquad (2.37)$$

Since $\Theta = 0$, (2.37) reduces to

$$\nabla^2\tau_{ij} = 0$$

which gives

$$\nabla_1^2\tau_{13} = \nabla_1^2\tau_{23} = 0$$

where $\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

So we have $\frac{\partial}{\partial x_2}(\nabla_1^2\Phi) = 0 = \frac{\partial}{\partial x_1}(\nabla_1^2\Phi)$

$$\therefore \nabla_1^2\Phi = \text{constant} = c, \text{ say} \quad (2.38)$$

Now

$$\vec{\nabla}\omega_3 = \tfrac{1}{2}\vec{\nabla}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

$$= \hat{k}\frac{\partial}{\partial x_1}e_{23} = \hat{k}\frac{\partial}{\partial x_2}e_{13}$$

$$= -\hat{k}\frac{\alpha}{2}c \text{ (by 2.38)}$$

so that

$$d\omega_3 = \vec{\nabla}\omega_3 . d\vec{r} = -\tfrac{1}{2}c\ \alpha dx_3$$

i.e. $\omega_3 = -\tfrac{1}{2}c\ \alpha dx_3 + \text{constant}$

since x_3=0 plane is rigidly fixed, so the constant =0 and hence

$$\omega_3 = -\tfrac{1}{2}c\ \alpha dx_3 .$$

If we identify the twist per unit length about x_3-axis by α, then

$$-\tfrac{1}{2}c\alpha = \frac{\partial\omega_3}{\partial x_3} = \alpha \quad (2.39)$$

i.e. $c = -2$ (2.40)

and, therefor, $\omega_3 = \alpha x_3$

On the free lateral surface of the rod we have

$$\nu_1\tau_{31} + \nu_2\tau_{32} = 0 \quad (2.41)$$

where $\nu_1 = \frac{dx_2}{ds},\ \nu_2 = -\frac{dx_1}{ds}$

So we have

$\frac{d\Phi}{ds} = 0$, i.e., Φ = constant on the lateral surface.

Thus the stress function ϕ satisfies the following Dirichlet problem,

$$\left.\begin{array}{l}\nabla_1^2\Phi = -2 \text{ within the cross-section } S.\\ \text{and } \Phi = \text{constant on the boundary } L \text{ of } S.\end{array}\right\} \quad (2.42)$$

ϕ is known as **Prandtl's stress function**

The resultant end moment M given by (2.32) can be simplified as

$$M = -\mu\alpha\int_S \left(x_1\Phi_{,x_1} + x_2\Phi_{,x_2}\right)ds$$

$$= -\mu\alpha\int_S \left(\frac{\partial}{\partial x_1}(x_1\Phi) + \frac{\partial}{\partial x_2}(x_2\Phi)\right) ds + 2\mu\alpha\int_S \Phi ds$$

$$= -\mu\alpha\int_L \left(\nu_1 x_1 + \nu_2 x_2\right)\Phi ds + 2\mu\alpha\int_S \Phi ds$$

$$0 + 2\mu\alpha\int_S \Phi ds \quad [\because \Phi = \text{constant} = 0 \text{ on } L]$$

$$\text{i.e., } D = \frac{M}{\alpha} = 2\mu\int_S \Phi ds \tag{2.43}$$

D is known as **torsional rigidity** which gives the measure of the applied torque to produce unit angle of twist.

Now,

$$\vec{\nabla}\omega_1 = \tfrac{1}{2}\vec{\nabla}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$

$$= \frac{\alpha}{2}\left[\hat{i}\Phi_{,22} - \hat{j}\Phi_{,12}\right]$$

$$d\omega_1 = \vec{\nabla}\omega_1 . d\vec{r}$$

$$= \frac{\alpha}{2}\left[\Phi_{,22}dx_1 - \Phi_{,12}dx_2\right]$$

$$= \frac{\alpha}{2}\left[\left(-2 - \Phi_{,11}\right)dx_1 - \Phi_{,12}dx_2\right] \text{ (using (2.42))}$$

$$\text{or, } d\omega_1 = -\alpha dx_1 - \frac{\alpha}{2}d\left(\Phi_{,1}\right)$$

Similarly

$$d\omega_2 = -\alpha dx_2 - \frac{\alpha}{2}d\left(\Phi_{,2}\right)$$

Integrating and neglecting the constant terms for rigid rotation

$$\omega_1 = -\alpha x_1 - \frac{\alpha}{2}\Phi_{,1} = -\alpha x_1 + e_{12} \tag{2.44}$$

$$\omega_2 = -\alpha x_2 - \frac{\alpha}{2}\Phi_{,2} = -\alpha x_2 + e_{13} \tag{2.45}$$

The displacement components $u_i\,(i = 1,2,3)$ can be obtained as follows

$$du_i = u_{i,j}dx_j = \left(e_{ij} + w_{ij}\right)dx_j \quad (i, j, = 1,2,3)$$

$$\begin{aligned}
\therefore du_1 &= e_{11}dx_1 + \left(e_{12} + w_{12}\right)dx_2 + \left(e_{13} + w_{13}\right)dx_3 \\
&= \left(e_{12} - w_3\right)dx_2 + \left(e_{13} + w_2\right)dx_3 \\
&= -\alpha x_3 dx_2 - \alpha x_2 dx_3 \qquad \text{(using (2.40) and (2.45))} \\
&= -\alpha\, d\left(x_2 x_3\right).
\end{aligned}$$

$$\begin{aligned}
du_2 &= \left(e_{21} + w_{21}\right)dx_1 + e_{33}dx_2 + \left(e_{23} + w_{23}\right)dx_3 \\
&= w_3 dx_1 + \left(e_{23} - w_1\right)dx_3 \\
&= \alpha\, d\left(x_1 x_3\right).
\end{aligned}$$

$$\begin{aligned}
du_3 &= \left(e_{31} + w_{31}\right)dx_1 + \left(e_{32} + w_{32}\right)dx_2 + e_{33}dx_3 \\
&= \left(e_{31} - w_2\right)dx_1 + \left(e_{32} + w_1\right)dx_2 \\
&= \alpha\left(x_2 + \Phi_{,2}\right)dx_1 - \alpha\left(x_1 + \Phi_{,1}\right)dx_2 \\
&= \alpha d\,\Phi\left(x_1, x_2\right)
\end{aligned}$$

provided

$$\frac{\partial}{\partial x_2}\left(x_2 + \Phi_{,2}\right) = \frac{\partial}{\partial x_1}\left(-x_1 - \Phi_{,1}\right)$$

or $\nabla_1^2\Phi = -2$ which is true.

Then integrating and neglecting rigid body rotation, we get

$$u_1 = -\alpha x_1 x_2, \; u_2 = \alpha x_1 x_3, \; u_3 = \alpha\phi\left(x_1, x_2\right) \tag{2.46}$$

$\phi\left(x_1, x_2\right)$ is known as **Saint-Venant torsion function,** where

$$\phi_{,1} = x_2 + \Phi_{,2} \tag{2.47}$$

$$\phi_{,2} = -x_1 - \Phi_{,1}$$

Hence, $\tau_{13} = \mu\alpha\Phi_{,2} = -\mu\alpha\left(x_2 - \phi_{,1}\right)$ (2.48)

$$\tau_{23} = -\mu\alpha\Phi_{,1} = \mu\alpha\left(x_1 + \phi_{,2}\right)$$

To find the equation satisfied by $\phi(x_1, x_2)$, we eliminate Φ, and get

$$\nabla_1^2\phi = 0 \tag{2.49}$$

The boundary condition (2.41) is given in terms of ϕ as

$$\nu_1\tau_{31} + \nu_3\tau_{32} = 0 \quad \text{on } L$$

i.e. $\dfrac{dx_2}{ds}\left(-x_2 + \phi_{,1}\right) + \left(-\dfrac{dx_1}{ds}\right)\left(-x_1 + \phi_{,2}\right) = 0$ on L

or $\phi_{,1}\dfrac{dx_2}{ds} + \phi_{,2}\left(-\dfrac{dx_1}{ds}\right) = x_1\dfrac{dx_1}{ds} + x_2\dfrac{dx_2}{ds}$ on L

$$\frac{\partial\phi}{\partial n} = \frac{\partial}{\partial s}\left(\frac{r^2}{2}\right) \quad \text{on } L \tag{2.50}$$

Thus the torsion function ϕ satisfies the following Neumann problem

$$\nabla_1^2\phi = 0 \quad \text{in } S \tag{2.51}$$

$$\frac{\partial\phi}{\partial n} = \frac{rdr}{ds} \quad \text{on } L$$

Let ψ be a harmonic function conjugate to ϕ; then ϕ and ψ must satisfy Canchy-Riemann differential equations

$$\phi_{,1} = \psi_{,2}$$

$$\phi_{,2} = \psi_{,1} \tag{2.52}$$

so that

$$\frac{r\,dr}{ds} = \frac{\partial\phi}{\partial n} = \nu_1\phi_1 + \nu_2\phi_2$$

$$= \psi_{,2}\frac{dx_2}{ds} + \left(-\psi_{,1}\right)\left(-\frac{dx_1}{ds}\right)$$

$$= \frac{d\psi}{ds}$$

$$\therefore \psi = \frac{r^2}{2} + \text{constant on } L \tag{2.53}$$

Hence, the conjugate torsion function ψ satisfies the following Dirichlet type problem

$$\nabla_1^2 \psi = 0 \quad \text{in } S$$
$$\psi = \frac{r^2}{2} + \text{constant} \quad \text{on } L \tag{2.54}$$

Stress components in terms of ψ:

$$\tau_{13} = \mu\alpha\left(\psi_{,2} - x_2\right)$$
$$\tau_{23} = \mu\alpha\left(-\psi_{,1} + x_1\right) \tag{2.55}$$

Relation between ϕ and ψ:

$$\begin{aligned} d\psi &= \psi_{,1}dx_1 + \psi_{,2}dx_2 \\ &= -\phi_{,2}dx_1 + \phi_{,1}dx_2 \\ &= \left(x_1 + \Phi_{,1}\right)dx_1 + \left(x_2 + \Phi_{,2}\right)dx_2 \\ &= d\Phi + d\left(\frac{x_1^2 + x_2^2}{2}\right) \end{aligned}$$

$$\therefore \psi = \Phi + \frac{r^2}{2} \tag{2.56}$$

Torque M in terms of ϕ:

$$\begin{aligned} M &= \int_S \left(x_1\tau_{23} - x_2\tau_{13}\right)ds \\ &= \mu\alpha\int_S \left(-x_1\Phi_{,1} - x_2\Phi_{,2}\right)ds \\ &= \mu\alpha\int_S \left(x_1^2 + x_2^2\right)ds + \mu\alpha\int_S \left(x_1\Phi_{,2} - x_2\Phi_{,1}\right)ds \\ &= \mu\alpha\int_S r^2 ds + \mu\alpha\int_S \left(-\frac{\partial}{\partial x_1}(x_2\phi) + \frac{\partial}{\partial x_2}(x_1\phi)\right)ds \\ &= \mu\alpha\int_S r^2 ds - \mu\alpha\int_L \phi\left(x_2 dx_2 + x_1 dx_1\right) \qquad \text{[using Green's theorem]} \end{aligned}$$

$$= \mu\alpha \int_S r^2 ds + \frac{\mu\alpha}{2} \int_L r^2 \frac{\partial \phi}{\partial s} ds \quad (2.57)$$

Circulation of Stress along the boundary :

Circulation of stress vector $\bar{F}_3$ along the boundary L

$$= \int_L (\tau_{13} dx_1 + \tau_{23} dx_2)$$

$$= \mu\alpha \int_L (\Phi_{,2} dx_1 - \Phi_{,1} dx_2)$$

$$= \mu\alpha \int_L \left(-\Phi_{,2} \left(\frac{-dx_1}{ds} \right) - \Phi_{,1} \left(\frac{dx_2}{ds} \right) \right) ds$$

$$= -\mu\alpha \int_L \bar{\nabla}_1 \Phi . \hat{n} ds = -\mu\alpha \int_S \nabla_1^2 \Phi ds$$

$$= 2\mu\alpha S \quad (\because \nabla_1^2 \Phi = -2) \quad (2.58)$$

Lines of shear stress :

The stress vector on the area x_3= constant of a rod in torsion is equal to

$$\bar{T}_3 = \hat{i}\tau_{31} + \hat{j}\tau_{23}$$

$$= \mu\alpha \left[\hat{i}\Phi_{,2} - \hat{j}\Phi_{,1} \right]$$

Therefore

$$\bar{T}_3 . \bar{\nabla}\Phi = 0 .$$

That is $\bar{T}_3$ at any point of cross-section is directed tangentially to the curve $\Phi(x_1, x_2)$ = constant passing through that point, i.e.

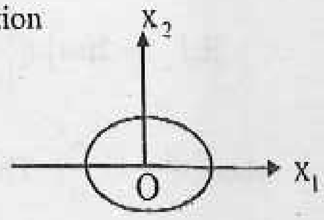
$$\bar{T}_3 = |T_3| \bar{t}$$

The family of the closed curves, defined in the plane of cross-section, are for this region, called the **lines of shear stress**.

2.6 Solution of the Torsion Problem for Certain Particular Cases

(i) Rod of elliptic cross-section :

Let us choose x_1 and x_2 axes along the axes of the elliptic cross-section. The profile of the cross-section of the rod in torsion is given by the equation



$$L: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0$$

The Prandlt stress function Φ satisfies the boundary value problem :

$$\nabla_1^2 \Phi = -2 \text{ in } S \tag{2.62}$$

$$\Phi = 0 \quad \text{on } L \tag{2.63}$$

Satisfying the boundary condition (2.63), we get,

$$\Phi = A\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right)$$

where A is a suitable constant and is determined by equation (2.62) as

$$A = -\frac{a^2 b^2}{a^2 + b^2}$$

therefore

$$\Phi = -\frac{a^2 b^2}{a^2 + b^2}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) \tag{2.64}$$

The stress components are

$$\tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = 0$$

$$\tau_{13} = \mu\alpha\Phi_{,2} = 2\mu\alpha A \frac{x_2}{b^2} \tag{2.65}$$

$$\tau_{23} = -\mu\alpha\Phi_{,1} = -2\mu\alpha A \frac{x_1}{a^2}$$

The resultant shearing stress is

$$\tau = \sqrt{\tau_{13}^2 + \tau_{23}^2} = 2\mu\alpha A \sqrt{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}}.$$

On the boundary L,

$$\tau = 2\mu\alpha A \sqrt{\frac{x_1^2}{a^4} + \frac{1}{b^2}\left(1 - \frac{x_1^2}{a^2}\right)}$$

To find τ_{max} in the range $-a < x < a$, we have

$$|\bar{\tau}_3|_{max} = 2\mu\alpha|A|\left\{\frac{1}{a^2}+\frac{e^2x_2^2}{b^4}\right\}^{\frac{1}{2}}$$

where $e^2 = 1-\frac{b^2}{a^2}$. Thus the maximum value occurs at $x_2 = \pm e$. Therefore,

$$|\bar{\tau}_3|_{max} = 2\mu\alpha|A|\left(\frac{1}{a^2}+\frac{e^2}{b^2}\right)^{1/2}$$

$$= 2\mu\alpha\frac{|A|}{b}.$$

The torsional rigidity $D = \frac{M}{\alpha} = 2\mu\int_S \Phi ds$

$$= 2\mu A\int_S\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}-1\right)ds$$

$$= 2\mu A.\pi ab\left[\frac{1}{a^2}\frac{a^2}{4}+\frac{1}{b^2}\frac{b^2}{4}-1\right]$$

$$= \pi\mu\frac{a^3b^3}{a^2+b^2} \quad \text{(using the value of } A\text{)} \qquad (2.66)$$

To find ϕ we use the relation

$$\phi_1 = x_2+\Phi_2 = \frac{b^2-a^2}{a^2+b^2}.x_2$$

$$\phi_2 = -x_1-\Phi_1 = -\frac{a^2-b^2}{a^2+b^2}.x_1$$

Therefore,

$$d\phi = -\frac{a^2-b^2}{a^2+b^2}(x_2dx_1+x_1dx_2)$$

so, $\phi = -\frac{a^2-b^2}{a^2+b^2}x_1x_2$

and $\psi = \Phi+\frac{1}{2}r^2$

$$= \frac{a^2-b^2}{2(a^2+b^2)}x_1^2+\frac{b^2-a^2}{2(a^2+b^2)}x_2^2+\frac{a^2b^2}{a^2+b^2} \qquad (2.68)$$

As a particular case, we may obtain the torsion problem of a rod of circular cross-section from the above by putting $b = a$:

Then

$$\Phi = -\tfrac{1}{2}\left(x_1^2 + x_2^2 - a^2\right) \tag{2.69}$$

The displacement component are

$$u_1 = -\alpha x_2 x_3,\ u_2 = \alpha x_1 x_3,\ u_3 = \alpha\phi = 0$$

since from (2.67) for $a = b, \phi = 0$.

The torsional rigidity $= D = \dfrac{M}{\alpha} = 2\mu\int_S \Phi ds$

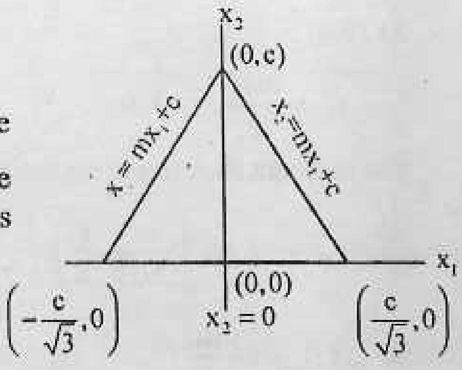
$$= \frac{\pi\mu a^4}{2}.$$

The resultant shearing stress at an arbitrary point in the rod is given by

$$|\vec{\tau}_3| = \mu\alpha r$$

where $r = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}$ is the distance from the axis of the rod. This stress acts in the direction of the tangent to the circle of radius r and is constant on this curve

$$|\vec{\tau}_3|_{max} = \mu\alpha a .$$



(2) Rod of Equilateral Triangular Cross-section :

Consider the cross-section of a bar bounded by the straight lines $x_2 = 0$ and $x_{-2} = \pm mx_1 + c$. Prandtl stress function Φ satisfies the boundary value problem.

$\nabla_1^2\Phi = -2$ within the cross-section (2.70)

and $\Phi = 0$ on the boundary $L : x_2\left[\left(x_2 - c\right)^2 - m^2 x_1^2\right] = 0$ (2.71)

Satisfying the boundary condition (6.71), let us take

$\Phi = Ax_2\left[\left(x_2 - c\right)^2 - m^2 x_1^2\right]$ within S (2.72)

Now,

$\nabla_1^2\Phi = -2$ demand that

$$A\left[\left(6-2m^2\right)x_2-4c\right]=-2 \quad (2.73)$$

Since (2.73) holds for any point within S, so

$$A\left(6-2m^2\right)=0 \text{ and } 4AC=2$$

$$\therefore m=\pm\sqrt{3}, \text{ and } A=\frac{1}{2c}$$

Hence, $$\Phi=\frac{1}{2c}x_2\left[\left(x_2-c\right)^2-3x_1^2\right] \quad (2.74)$$

Now,

$$\tau_{13}=\mu\alpha\Phi_2=\frac{\mu\alpha}{2c}\left[3x_2^2-4x_2c+c^2-3x_1^2\right]$$

$$\tau_{23}=-\mu\alpha\Phi_1=-\frac{\mu\alpha}{2c}\left(-6x_2x_1\right)$$

At (0,0)

$$\tau_{13}=\frac{\mu\alpha c}{2},\ \tau_{23}=0$$

The resultant shearing stress has a maximum value at the mid points of the sides of the triangle, i.e. at (0,0), $\left(\frac{-c}{2\sqrt{3}},\frac{c}{2}\right),\left(\frac{c}{2\sqrt{3}},\frac{c}{2}\right)$ and it is

$$|\vec{\tau}_3|_{max}=\frac{\mu\alpha c}{2}$$

where c is the altitude of the equlateral triangle.

On $x_2=0$,

$$\tau_{13}=\frac{\mu\alpha}{2c}\left[c^2-3x_1^2\right]=0 \text{ when } x_1=\pm\frac{c}{\sqrt{3}}$$

$$\tau_{23}=0$$

and at $(0,\ c)$

$$\tau_{13}=\frac{\mu\alpha}{2c}\times 0=0$$

$$\tau_{23}=0$$

The stress components vanishes at the vertices and at the centre of the section Torsional rigidity is

$$D = \frac{M}{\alpha} = 2\mu \int_S \Phi\, ds$$

$$= \frac{2\mu}{2c} 2 \int_0^c dx_1 \int_0^{-\sqrt{3}x_1 + c} x_2 \left\{ (x_2 - c)^2 - 3x_1^2 \right\} dx_2$$

$$= \frac{\mu c^4}{15\sqrt{3}}$$

and

$$\phi = x_1 \left[3x_2^2 - x_1^2 + c^2 - 2cx_2 \right] = x_1 \left[2x_2^2 - x_1^2 + (x_2 - c)^2 \right]$$

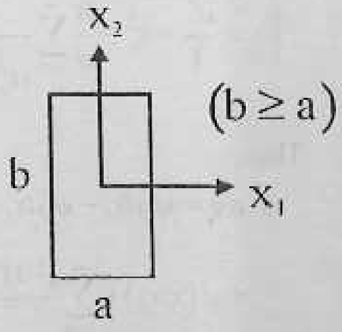
$$\psi = \Phi + \frac{x_1^2 + x_2^2}{2}$$

(3) A rod of rectangular Cross-section :

We take the origin at the centre of the rectangular section of sides a and b $(\geq a)$ and the coordinate axes parallel to the sides as shown in the figure. We solve the problem in terms of conjugate torsion function ψ satisfying

$$\nabla_1^2 \psi = 0 \text{ in S}$$

and $\psi = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ on $L : \left(x_1^2 - \frac{a^2}{4} \right)\left(x_2^2 - \frac{b^2}{4} \right) = 0$



Let us set

$$\psi = \frac{a^2}{4} - x_1^2 + \frac{x_1^2 + x_2^2}{2} + \psi_1(x_1, x_2)$$

where

$$\nabla_1^2 \psi_1 = 0 \text{ in S}$$

and $\psi_1 = x_1^2 - \frac{a^2}{4}$ on L

Now ψ and so ψ_1 must be an even function in both the variables x_1 and x_2 and

$\psi_1\left(\pm \frac{a}{2}, x_2 \right) = 0$ on L

Therefore, we have the solution.

$$\psi_1 = \sum_0^{\infty} a_n \cosh(\alpha_n n_2)\cos(\alpha_n x_1), \text{ where } \frac{a\alpha_n}{2} = (2n+1)\frac{\pi}{2}$$

Now, $\psi_1\left(x_1, \pm\frac{b}{2}\right) = x_1^2 - \frac{a^2}{4} = \sum_{n=0}^{\infty} a_n \cosh\frac{\alpha_n b}{2}\cos\alpha_n x_1$

$$\therefore a_j = \frac{8(-1)^{j+1}}{a\alpha_j^3 \cosh\alpha_j \frac{b}{2}}$$

Hence,

$$\psi_1 = \sum_{n=0}^{\infty} \frac{8(-1)^{n+1}}{a\alpha_n^3 \cosh\left(\frac{\alpha_n b}{2}\right)} \cosh(\alpha_n x_2)\cos(\alpha_n x_1)$$

Therefore,

$$\psi = \frac{a^2}{4} + \frac{x_2^2 - x_1^2}{2} + \psi_1(x_1, x_2)$$

and so

$$\Phi = \psi - \frac{x_1^2 + x_2^2}{2}$$

$$= \frac{a^2}{4} - x_1^2 - \frac{8}{a}\sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha_n^3 \cosh\left(\frac{\alpha_n b}{2}\right)} \cosh(\alpha_n x_2)\cos(\alpha_n x_1)$$

Thus,

$$d\phi = \psi_2 dx_1 - \psi_1 dx_2$$

$$= d(x_1 x_2) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} d(\sinh(\alpha_n x_2)\sin(\alpha_n x_1))}{a\,\alpha_n^3 \cosh\left(\alpha_n \frac{b}{2}\right)}$$

Therefore, neglecting a non-essential constant

$$\phi = x_1 x_2 - \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty} \frac{(-1)^3 \sinh(\alpha_n x_2)\sin(\alpha_n x_1)}{(2n+1)^3 \cosh\left(\frac{\alpha_n b}{2}\right)}.$$

The shearing stresses are

$$\tau_{13} = \mu\alpha\Phi_2 = -\mu\alpha\frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n \sinh(\alpha_n x_2)\cos(\alpha_n x_1)}{(2n+1)^2\cosh\left(\frac{\alpha_n b}{2}\right)}$$

$$\tau_{23} = -\mu\alpha\Phi_1 = \mu\alpha\left(2x_1 - \frac{8a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n \cosh(\alpha_n x_2)\sin(\alpha_n x_1)}{(2n+1)^2\cosh\left(\frac{\alpha_n b}{2}\right)}\right)$$

The torsional rigidity is

$$D = \frac{M}{\alpha} = 2\mu\int_S \Phi ds$$

$$= \frac{\mu b a^3}{3}\left(1 - \frac{192a}{\pi^5 b}\sum_{n=0}^{\infty}\frac{\tanh\left(\frac{\alpha_n b}{2}\right)}{(2n+1)^5}\right).$$

2.7 Summary

In this unit, some problems of extension and torsion of a beam of uniform cross-section have been considered. Axial extension, stretching of beam by its own weight and bending of beam by terminal couples have been discussed. The torsion of cylindrical having different cross-sections have also been studied.

2.8 Excercises

1. Short Questions :

(a) Define central line of the beam.

(b) Obtain the displacement components given in (equation number 2.29).

(c) Define torsional rigidity and obtain the expression in terms of Prandtl stress function.

(d) Find the circulation of stress along the boundary of the prismatic bar.

(e) What is meant by the line of shear stress?

2. ***Broad answer type :***

(a) Find the displacement component in an elastic beam when it is under axial extension.

(b) Obtain the stress, strain components in an elastic beam when it is under bending by terminal couples.

(c) Show that the problem of torsion of a long prismatic rod of cross-sections of isotropic material twisted by end is equivalent to the boundary value problem $\nabla_1^2\Phi = -2, \ (x_1, x_2) \in S, \ \nabla_1^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and $\Phi =$ const. on the boundary L, $S + L$ is a normal cross-section of the rod and Φ is the Prandtl's stress function.

(d) Show that the maximum resultant shearing stress which arises in the torsion problem occurs on the lateral surface of the rod.

(e) Solve the torsion problem of a long prismatic bar of elliptic cross-section twisted by end couples. Also find the maximum resultant shearing stress of the bar.

(f) Show that in the torsion problem of long prismatic bar of equilateral triangular cross-section twisted by end couple the torsional rigidity is given by $\frac{\mu c^4}{15\sqrt{3}}$.

(g) Find the shearing stresses in the torsion problem of long prismatic bar of rectangular cross-section twisted by end couples.

Unit-3 □ Semi-Infinite Solids With Prescribed Displacements or Stresses on The Boundary.

Structure

(In this unit we shall use unabridged notation instead of tensor notations)

3.1 Semi-infinite Solid with Prescribed Displacements on the Plane Boundary

Suppose the bounding surface $z = 0$ of the semi-infinite solid occupying the space $z \geq 0$ has prescribed displacements

$$u = U\left(x^1, y^1\right), v = V\left(x^1, y^1\right), w = W\left(x^1, y^1\right).$$

The Navier equations of equilibrium in the absence of body forces are of the type

$$(\lambda + \mu)\frac{\partial \vartheta}{\partial x} + \mu \nabla^2 u = 0$$

i.e. $$\nabla^2 u = -\frac{\lambda + \mu}{\mu}\frac{\partial \vartheta}{\partial x} = -\frac{1}{1-2\sigma}\frac{\partial \vartheta}{\partial x},$$

where $\sigma = \dfrac{\lambda}{2(\lambda + \mu)}$ is the Poisson's ratio. Now, in the absence of body force, ϑ is a harmonic function and so we can take

$$\vartheta = -2(1-2\sigma)\frac{\partial F}{\partial z}$$

where F is a harmonic function. Then

$$\nabla^2 u = 2\frac{\partial^2 F}{\partial x \partial z} = \nabla^2\left(z\frac{\partial F}{\partial x}\right)$$

Similarly, $\nabla^2 v = 2\frac{\partial^2 F}{\partial y \partial z} = \nabla^2\left(z\frac{\partial F}{\partial y}\right)$ and $\nabla^2 w = 2\frac{\partial^2 F}{\partial z^2} = \nabla^2\left(z\frac{\partial F}{\partial z}\right)$.

Hence we have

$$u = z\frac{\partial F}{\partial x} + \phi_1,\ v = z\frac{\partial F}{\partial y} + \phi_2,\ w = z\frac{\partial F}{\partial z} + \phi_3$$

where ϕ_1, ϕ_2, ϕ_3 are harmonic functions whose values on the plane $z = 0$ are U, V, W respectively.

3.2 Semi-infinite Solid with Prescribe Surface Tractions on the Plane Boundary

Suppose the plane boundary $z = 0$ of the semi-infinite solid occupying the space $z \geq 0$ be subjected to prescribed stresses

$$\tau_{xz} = -X(x', y'),\ \tau_{yz} = -Y(x', y'),\ \tau_{zz} = -Z(x', y').$$

Now, in the absence of body forces, Beltrami-Michell compatibility relations are of the type

$$\nabla^2 \tau_{xz} + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x \partial z} = 0$$

where $\Theta = \tau_{xx} + \tau_{yy} + \tau_{zz}$ is harmonic function. Noting that

$$2\frac{\partial^2 \Theta}{\partial x \partial z} = \nabla^2\left(z\frac{\partial \Theta}{\partial x}\right),$$

we have from the above equation

$$\nabla^2\left[\tau_{xz} + \frac{1}{2(1+\sigma)} . z\frac{\partial \Theta}{\partial x}\right] = 0$$

so that $\tau_{xz} = -\frac{1}{2(1+\sigma)} . z\frac{\partial \Theta}{\partial x} + \psi_1$

similarly, $\tau_{yz} = -\frac{1}{2(1+\sigma)} . z\frac{\partial \Theta}{\partial y} + \psi_2$ (3.1)

$$\tau_{zz} = -\frac{1}{2(1+\sigma)} \cdot z\frac{\partial \Theta}{\partial z} + \psi_3 ,$$

where ψ_1, ψ_2, ψ_3 are harmonic functions such that on the surface $z=0$

$$\psi_1 = \tau_{xz} = -X, \psi_2 = \tau_{yz} = -Y, \psi_3 = \tau_{zz} = -Z .$$

Now, using (3.1), the equation of equilibrium

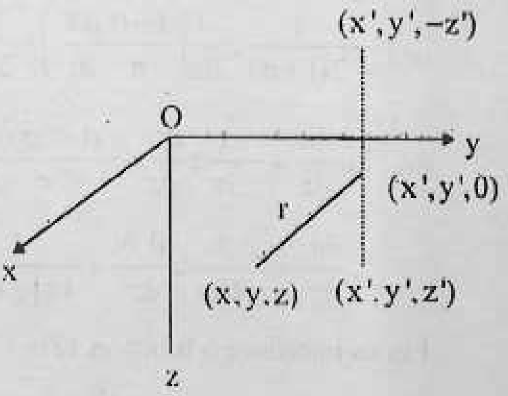
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = 0$$

gives $-\frac{1}{2(1+\sigma)}\left[z\left(\nabla^2\Theta\right) + \frac{\partial \Theta}{\partial z}\right] + \left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right) = 0$

i.e. $-\frac{1}{2(1+\sigma)} \frac{\partial \Theta}{\partial z} + \left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right) = 0 \quad \left(\because \nabla^2 \quad = 0\right)$ (3.2)

Let $L = \iint \frac{X(x^1, y^1)}{r} dx^1 dy^1$, $M = \iint \frac{Y(x^1, y^1)}{r} dx^1 dy^1$,

$$N = \iint \frac{Z(x^1, y^1)}{r} dx^1 dy^1$$



where $r^2 = \left(x - x^1\right)^2 + \left(y - y^1\right)^2 + z^2$.

$$\therefore \left(\frac{\partial L}{\partial z}\right)_{z=0} = -2\pi X = 2\pi\left(\psi_1\right)_{z=0}$$

so that we can take $\psi_1 = \frac{1}{2\pi} \frac{\partial L}{\partial z}$.

Similarly, $\psi_2 = \frac{1}{2\pi} \frac{\partial M}{\partial z}$ and $\psi_3 = \frac{1}{2\pi} \frac{\partial N}{\partial z}$.

So from (3.2) we have

$$\frac{\partial \Theta}{\partial z} = \frac{1+\sigma}{\pi} \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right)$$

i.e. $\Theta = \frac{1+\sigma}{\pi}\left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right)$ (3.3)

Thus the distribution of stress in the interior of the body is known in terms of L,M,N.

Distributed normal pressure on the plane $z = 0$.

Suppose, on the plane $z = 0$,

$$\tau_{zx} = 0, \tau_{yz} = 0, \tau_{zz} = -Z(x', y').$$

Then $\psi_1 = \psi_2 = 0$ and $\psi_3 = \frac{1}{2\pi}\frac{\partial N}{\partial z}$ and by (3.3), $\Theta = \frac{1+\alpha}{\pi}\frac{\partial N}{\partial z}$.

Let $\chi = \iint Z \log(z+r) dx'dy'$, so that $\frac{\partial \chi}{\partial z} = \iint \frac{Z}{r} dx'dy'$ on $z = 0$ and hence

$$N = \frac{\partial \chi}{\partial z}.$$

Now, $\tau_{zz} = \lambda\vartheta + 2\mu\frac{\partial w}{\partial z}$

$$\text{or,} \quad -\frac{1}{2(1+\sigma)} z\frac{\partial \Theta}{\partial z} + \psi_3 = \frac{\lambda\Theta}{3\lambda+2\mu} + 2\mu\frac{\partial w}{\partial z} \quad (\because \ = (3\lambda+2\mu)\vartheta)$$

$$\text{or,} \quad -\frac{1}{2(1+\sigma)} z\frac{\partial}{\partial z}\left(\frac{1+\sigma}{\pi}\frac{\partial N}{\partial z}\right) + \frac{1}{2\pi}\frac{\partial N}{\partial z} = \frac{\sigma}{1+\sigma}\cdot\frac{1+\sigma}{\pi}\frac{\partial N}{\partial z} + 2\mu\frac{\partial w}{\partial z} \ (\text{by}(3.1))$$

$$\text{or,} \quad 2\mu\frac{\partial w}{\partial z} = -\frac{1}{2\pi} z\frac{\partial^2 N}{\partial z^2} + \frac{(1-2\sigma)}{2\pi}\frac{\partial N}{\partial z}$$

$$\therefore \frac{\partial w}{\partial z} = -\frac{1}{4\pi\mu} z\frac{\partial^2 N}{\partial z^2} + \frac{1}{4\pi(\chi+\mu)}\frac{\partial N}{\partial z}. \qquad (3.4)$$

Let us introduce a function Ω by

$$\Omega = -\frac{zN}{4\mu\pi} - \frac{\chi}{4\pi(\lambda+\mu)}$$

Then

$$\frac{\partial \Omega}{\partial z} = -\frac{N}{4\mu\pi} - \frac{1}{4\mu\pi} z\frac{\partial M}{\partial z} - \frac{1}{4\pi(\lambda+\mu)}\frac{\partial \chi}{\partial z} = -\frac{N}{4\mu\pi} - \frac{1}{4\mu\pi} z\frac{\partial N}{\partial z} - \frac{N}{4\pi(\lambda+\mu)}$$

$$\text{i.e.} \quad \frac{\partial \Omega}{\partial z} = -\frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)} N - \frac{1}{4\pi\mu} z\frac{\partial N}{\partial z} \qquad (3.5)$$

Also

$$\frac{\partial^2\Omega}{\partial z^2}=-\frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\frac{\partial N}{\partial z}-\frac{1}{4\pi\mu}\frac{\partial N}{\partial z}-\frac{1}{4\pi\mu}z\frac{\partial^2 N}{\partial z^2}=-\frac{2\lambda+3\mu}{4\pi\mu(\lambda+\mu)}\frac{\partial N}{\partial z}-\frac{1}{4\pi\mu}z\frac{\partial^2 N}{\partial z^2}$$

i.e. $-\frac{1}{4\pi\mu}z\frac{\partial^2 N}{\partial z^2}=\frac{\partial^2\Omega}{\partial z^2}+\frac{2\lambda+3\mu}{4\pi\mu(\lambda+\mu)}\frac{\partial N}{\partial z}$

Hence from (3.4) we get

$$\frac{\partial w}{\partial z}=\frac{\partial^2\Omega}{\partial z^2}+\frac{2\lambda+3\mu}{4\pi(\lambda+\mu)\mu}\frac{\partial N}{\partial z}+\frac{1}{4\pi(\lambda+\mu)}\frac{\partial N}{\partial z}=\frac{\partial^2\Omega}{\partial z^2}+\frac{\lambda+2\mu}{2\pi(\lambda+\mu)\mu}\frac{\partial N}{\partial z}$$

so that on integration

$$w=\frac{\partial\Omega}{\partial z}+\frac{\lambda+2\mu}{2\pi(\lambda+\mu)\mu}N \tag{3.6}$$

Again, $\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$ gives

$$\frac{\partial u}{\partial z}=\frac{1}{\mu}\tau_{xz}-\frac{\partial w}{\partial x}=-\frac{1}{2\mu(1+\sigma)}z\frac{\partial\Theta}{\partial x}-\frac{\partial^2\Omega}{\partial x\partial z}-\frac{\lambda+2\mu}{2\pi(\lambda+\mu)\mu}\frac{\partial N}{\partial x}$$

(using (3.1) with $\psi_1=0$)

$$=-\frac{1}{2\mu(1+\sigma)}z\frac{\partial}{\partial x}\left(\frac{1+\sigma}{\pi}\frac{\partial N}{\partial z}\right)-\frac{\partial^2\Omega}{\partial x\partial z}-\frac{\lambda+2\mu}{2\pi(\lambda+\mu)\mu}\frac{\partial N}{\partial x}$$

(using (3.3) with $L=M=0$)

$$=2\frac{\partial}{\partial x}\left[-\frac{1}{4\pi\mu}z\frac{\partial N}{\partial z}-\frac{\lambda+2\mu}{2\pi(\lambda+\mu)\mu}N\right]-\frac{\partial^2\Omega}{\partial x\partial z}$$

$$=2\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial z}\right)-\frac{\partial^2\Omega}{\partial x\partial z}\quad\text{[by (3.5)]}$$

$$=\frac{\partial^2\Omega}{\partial x\partial z},$$

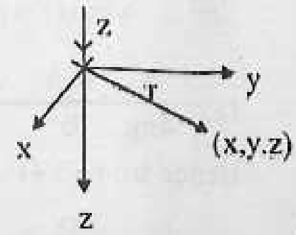
so that $u=\frac{\partial\Omega}{\partial x}$. Similarly, $v=\frac{\partial\Omega}{\partial y}$.

Thus, the displacement components for the present problem are

$$u=\frac{\partial\Omega}{\partial x},\ v=\frac{\partial\Omega}{\partial y},\ w=\frac{\partial\Omega}{\partial z}+\frac{\lambda+2\mu}{2\pi(\lambda+\mu)\mu}N \tag{3.7}$$

Normal pressure at a point on the plane boundary $z = 0$.

Suppose that on the plane $z = 0$,



$$\tau_{xz} = 0,\ \tau_{yz} = 0,\ \tau_{zz} = \begin{cases} -Z \text{ at the origin} \\ 0 \text{ elsewhere,} \end{cases}$$

where Z is constant. Then from (3.1), the distribution of stresses in the interior of the medium is given by

$$\tau_{xz} = -\frac{1}{2(1+\sigma)} z\frac{\partial\Theta}{\partial x},\ \tau_{yz} = -\frac{1}{2(1+\sigma)} z\frac{\partial\Theta}{\partial y},\ \tau_{zz} = -\frac{1}{2(1+\sigma)} z\frac{\partial\Theta}{\partial z} + \psi_3 \quad (3.8)$$

where $(\psi_3)_{z=0} = (\tau_{zz})_{z=0}$

Let $N = \iint \frac{Z(x',y')}{r} dx'dy' = \frac{z}{r}$

($\therefore$ for a concentrated force at the origin $r^2 = x^2 + y^2 + z^2$)

$$\therefore \frac{1}{2\pi}\left(\frac{\partial N}{\partial z}\right)_{z=0} = -(Z)_{z=0} = (\tau_{zz})_{z=0} = (\psi_3)_{z=0}$$

so that $\Psi_3 = \frac{1}{2\pi}\frac{\partial N}{\partial z} = \frac{Z}{2\pi}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$ (3.9)

The equation of equilibrium

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z} = 0$$

then gives with $\psi_1 = \tau_2 = 0$ from (3.1)

$$-\frac{1}{2(1+\sigma)} z\nabla^2\Theta - \frac{1}{2(1+\sigma)}\frac{\partial\Theta}{\partial z} + \frac{\partial\psi_3}{\partial z} = 0,$$

i.e. $\frac{\partial\Theta}{\partial z} = 2(1+\sigma)\frac{\partial\psi_3}{\partial z}$ ($\because \nabla^2\Theta = 0$)

so that $\Theta = 2(1+\sigma)\psi_3 = \frac{(1+\sigma)Z}{\pi}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$ (by (3.9)) (3.10)

Now we have

$$\tau_{zz} = \lambda\vartheta + 2\mu\frac{\partial w}{\partial z} = \frac{\lambda}{3\lambda+2\mu}\Theta + 2\mu\frac{\partial w}{\partial z} \quad (\because \Theta = (3\lambda+2\mu)\vartheta)$$

or, $-\frac{1}{2(1+\sigma)} z \frac{\partial \Theta}{\partial z} + \psi_3 = \frac{\lambda}{3\lambda + 2\mu} . 2(1+\sigma)\psi_3 + 2\mu \frac{\partial w}{\partial z}$ (by 3.10)

$$2\mu \frac{\partial w}{\partial z} = -\frac{1}{2(1+\sigma)} z \frac{\partial}{\partial z}\left[2(1+\sigma)\psi_3\right] - \frac{\sigma}{1+\sigma} 2(1+\sigma)\psi_3 + \psi_3$$

$$\left[\because \sigma = \frac{\lambda}{2(\lambda+\mu)} \quad \frac{\sigma}{1+\sigma} = \frac{\lambda}{3\lambda + 2\mu}\right]$$

$$\therefore \frac{\partial w}{\partial z} = -\frac{1}{2\mu} z \frac{\partial \psi_3}{\partial z} + \frac{1-2\sigma}{2\mu}\psi_3 = -\frac{1}{2\mu} z \frac{\partial \psi_3}{\partial z} + \frac{\mu}{2\mu(\lambda+\mu)}\psi_3$$

$$= -\frac{1}{2\mu} z \frac{\partial}{\partial z}\left[\frac{Z}{2\pi}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right] + \frac{1}{2(\lambda+\mu)} . \frac{Z}{2\pi}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \frac{Z}{4\mu\pi} z \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) - \frac{Z}{4\pi(\lambda+\mu)} . \frac{z}{r^3} \quad \left[\begin{array}{l} \because r^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} \\ \because \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial z} = -\frac{z}{r^3} \end{array}\right]$$

$$= \frac{Z}{4\mu\pi} z \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) - \frac{Z}{4\pi(\lambda+\mu)} . \frac{z}{r^3}$$

$$= \frac{Z}{4\mu\pi}\left[\frac{\partial}{\partial z}\left(\frac{z^2}{r^3}\right) - \frac{z}{r^3}\right] - \frac{Z}{4\pi(\lambda+\mu)} . \frac{z}{r^3}$$

$$= \frac{Z}{4\mu\pi}\frac{\partial}{\partial z}\left(\frac{z^2}{r^3}\right) - \frac{(\lambda+2\mu)Z}{4\pi\mu(\lambda+\mu)} . \frac{z}{r^3}$$

$$= \frac{Z}{4\mu\pi}\frac{\partial}{\partial z}\left(\frac{z^2}{r^3}\right) + \frac{(\lambda+2\mu)Z}{4\pi\mu(\lambda+\mu)}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

so that, on integration, we have

$$w = \frac{Z}{4\mu\pi} . \frac{z^2}{r^3} + \frac{(\lambda+2\mu)Z}{4\pi\mu(\lambda+\mu)} . \frac{1}{r}$$

Again, $\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$

or, $-\frac{1}{2(1+\sigma)} z \frac{\partial \Theta}{\partial x} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$ (by (3.8))

$$\therefore \frac{\partial u}{\partial z} = -\frac{1}{2\mu(1+\sigma)} z \frac{\partial}{\partial x}[2(1+\sigma)\psi_3] - \frac{\partial w}{\partial x} \qquad \text{(by (3.10)}$$

$$= -\frac{1}{\mu} z \frac{\partial \psi_3}{\partial x} - \frac{\partial w}{\partial x}$$

$$= -\frac{1}{\mu} z . \frac{\partial}{\partial x}\left[\frac{Z}{2\pi} \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right] - \frac{\partial}{\partial x}\left[\frac{Z}{4\pi\mu} \frac{z^2}{r^3} + \frac{(\lambda+2\mu)Z}{4\pi\mu(\lambda+\mu)} . \frac{1}{r}\right]$$

$$= -\frac{Zz}{2\pi\mu} \frac{\partial}{\partial x}\left(-\frac{z}{r^3}\right) - \frac{Zz^2}{4\pi\mu} \frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) - \frac{(\lambda+2\mu)Z}{4\pi\mu(\lambda+\mu)} \frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= \frac{Zz^2}{2\pi\mu} \frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) - \frac{Zz^2}{4\pi\mu} \frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) - \frac{(\lambda+2\mu)Z}{4\pi\mu(\lambda+\mu)} \frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= \frac{Zz^2}{4\pi\mu}\left(-\frac{3}{r^4} . \frac{x}{r}\right) - \frac{(\lambda+2\mu)Z}{4\pi\mu(\lambda+\mu)}\left(-\frac{1}{r^2} . \frac{x}{r}\right)$$

$$= \frac{Z}{4\pi\mu}\left[\frac{\partial}{\partial z}\left(\frac{zx}{r^3}\right) - \frac{x}{r^3}\right] + \frac{(\lambda+2\mu)Z}{4\pi\mu(\lambda+\mu)} \frac{x}{r^3}$$

$$= \frac{Z}{4\pi\mu} \frac{\partial}{\partial z}\left(\frac{zx}{r^3}\right) + \frac{Z}{4\pi(\lambda+\mu)} . \frac{x}{r^3}$$

Noting that

$$\frac{\partial}{\partial z}\left\{\frac{1}{r(z+r)}\right\} = -\frac{1}{r^2} . \frac{z}{r} . \frac{1}{z+r} + \frac{1}{r} . \frac{1}{(z+r)^2} . \left(1 + \frac{z}{r}\right)$$

$$= -\frac{z}{r^3(z+r)} - \frac{1}{r^2(z+r)} = \frac{1}{r^3},$$

so that

$$\frac{\partial u}{\partial z} = \frac{z}{4\pi\mu} \frac{\partial}{\partial z}\left(\frac{zx}{r^3}\right) - \frac{z}{4\pi(\lambda+\mu)} \frac{\partial}{\partial z}\left\{\frac{x}{r(z+r)}\right\}$$

Integrating, we have

$$u = \frac{z}{4\pi\mu} . \frac{zx}{r^3} - \frac{z}{4\pi(\lambda+\mu)} . \frac{x}{r(z+r)}$$

Similarly, $v = \frac{z}{4\pi\mu}.\frac{yz}{r^3} - \frac{z}{4\pi(\lambda+\mu)}.\frac{y}{r(z+r)}$.

Hence the displacements an any point in the region $z > 0$ when a constant normal load z acts at the origin are given by

$$u = \frac{Z}{4\pi\mu}.\frac{zx}{r^3} - \frac{Z}{4\pi(\lambda+\mu)}.\frac{x}{r(z+r)},$$

$$v = \frac{Z}{4\pi\mu}.\frac{yz}{r^3} - \frac{Z}{4\pi(\lambda+\mu)}.\frac{y}{r(z+r)} \qquad (3.11)$$

$$w = \frac{Z}{4\pi\mu}.\frac{z^2}{r^3} + \frac{Z(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}.\frac{1}{r}$$

3.3 Simple Solutions

I. First Type :

Navier equations of equilibrium in the absence of body forces are

$$(\lambda+\mu)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\vartheta + \mu\nabla^2(u,v,w) = 0 \qquad (3.12)$$

It can easily be verified that the displacement given by

$$u = A.\frac{zx}{r^3},\ v = A.\frac{yz}{r^3},\ w = A.\left(\frac{z^2}{r^3} + \frac{\lambda+3\mu}{\lambda+\mu}.\frac{1}{r}\right), \qquad (3.13)$$

where A is constant and $r^2 = x^2 + y^2 + z^2$, satisfy the equations (3.12) except at the origin. Let us enclose the origin within a cavity of the body and calculate the traction across the surface of the cavity. The tractions corresponding with the displacements (3.13) over any surface are a system of forces in statical equilibrium by exclusion of the origin. Thus, in the case of the body with the cavity, the resultant force and the resultant moment of these tractions at the outer boundary are equal and opposite to those at the surface of the cavity. Now these tractions at the outerboundary do not depend on the shape or size of the cavity, so they may be calculated by taking the cavity to be spherical and taking the limit as the radius of the sphere diminishes indefinitely. We show that the displacements given by (3.13) is produced by a force $8\pi\mu(\lambda+2\mu)A/(\lambda+\mu)$ applied at the origin along the direction of the z-axis.

We rewrite (3.13) in the form

$$u=-A\frac{\partial^2 r}{\partial x\partial z},\ v=-A\frac{\partial^2 r}{\partial y\partial z},\ w=-A\left(\frac{\partial^2 r}{\partial z}-\frac{\lambda+2\mu}{\lambda+\mu}\nabla^2 r\right) \tag{3.14}$$

Then the cubical dilatation is

$$\vartheta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-A\frac{\partial^3 r}{\partial x^2\partial z}-A\frac{\partial^3 r}{\partial y^2\partial z}-A\frac{\partial^3 r}{\partial z^3}+A\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial}{\partial z}\left(\nabla^2 r\right)$$

$$=\frac{A\mu}{\lambda+\mu}\frac{\partial}{\partial z}\left(\nabla^2 r\right)=\frac{2A\mu}{\lambda+\mu}\frac{\partial r^{-1}}{\partial z} \qquad \left(\because \nabla^2 r=\frac{2}{r}\right)$$

Hence $\tau_{xx}=\lambda\vartheta+2\mu\dfrac{\partial u}{\partial x}=\dfrac{2A\lambda\mu}{\lambda+\mu}\dfrac{\partial r^{-1}}{\partial z}-2A\mu\dfrac{\partial^3 r}{\partial x^2\partial z}$

$$=\frac{2A\lambda\mu}{\lambda+\mu}\frac{\partial r^{-1}}{\partial z}-2A\mu\left[\frac{\partial r^{-1}}{\partial z}-3\frac{\partial r^{-1}}{\partial z}\left(\frac{\partial r}{\partial x}\right)^2\right]$$

i.e. $\tau_{xx}=2\mu A\dfrac{\partial r^{-1}}{\partial z}\left\{3\left(\dfrac{\partial r}{\partial x}\right)^2-\dfrac{\mu}{\lambda+\mu}\right\}.$

Similarly $\tau_{yy}=2\mu A\dfrac{\partial r^{-1}}{\partial z}\left\{3\left(\dfrac{\partial r}{\partial y}\right)^2-\dfrac{\mu}{\lambda+\mu}\right\},$ (3.15)

$$\tau_{zz}=2\mu A\frac{\partial r^{-1}}{\partial z}\left\{3\left(\frac{\partial r}{\partial z}\right)^2+\frac{\mu}{\lambda+\mu}\right\},$$

$$\tau_{yz}=2\mu A\frac{\partial r^{-1}}{\partial y}\left\{3\left(\frac{\partial r}{\partial z}\right)^2+\frac{\mu}{\lambda+\mu}\right\},$$

$$\tau_{zx}=2\mu A\frac{\partial r^{-1}}{\partial x}\left\{3\left(\frac{\partial r}{\partial z}\right)^2+\frac{\mu}{\lambda+\mu}\right\},$$

$$\tau_{xy}=6\mu A\frac{\partial r}{\partial x}.\frac{\partial r}{\partial y}.\frac{\partial r^{-1}}{\partial z}.$$

The tractions across any plane with normal $\bar{\nu}$ are given by

$$\overset{\vee}{T}_x = \tau_{xx}\cos(x,\nu)+\tau_{xy}\cos(y,\nu)+\tau_{xz}\cos(z,\nu)$$

$$= 2\mu A\frac{\partial r^{-1}}{\partial z}\left\{3\left(\frac{\partial r}{\partial z}\right)^2-\frac{\mu}{\lambda+\mu}\right\}\cos(x,\nu)+6\mu A\frac{\partial r}{\partial x}.\frac{\partial r}{\partial y}.\frac{\partial r^{-1}}{\partial z}\cos(y,\nu)$$

$$+2\mu A\frac{\partial r^{-1}}{\partial x}\left\{3\left(\frac{\partial r}{\partial z}\right)^2+\frac{\mu}{\lambda+\mu}\right\}\cos(z,\nu)$$

$$= 2\mu A\left[\frac{\mu}{\lambda+\mu}\left\{\cos(z,\nu)\frac{\partial r^{-1}}{\partial x}-\cos(x,\nu)\frac{\partial r^{-1}}{\partial z}\right\}\right.$$

$$\left.+3\frac{\partial r}{\partial x}.\frac{\partial r}{\partial z}\left\{-\frac{1}{r^2}\frac{\partial r}{\partial x}\cos(x,\nu)-\frac{1}{r^2}\frac{\partial r}{\partial y}\cos(y,\nu)-\frac{1}{r^2}\frac{\partial r}{\partial z}\cos(z,\nu)\right\}\right]$$

$$\therefore \overset{\vee}{T}_x = 2\mu A\left[\frac{\mu}{\lambda+\mu}\left\{\cos(z,\nu)\frac{\partial r^{-1}}{\partial x}-\cos(x,\nu)\frac{\partial r^{-1}}{\partial z}\right\}+3\frac{\partial r}{\partial x}\frac{\partial r}{\partial z}\frac{\partial r^{-1}}{\partial \nu}\right.$$

Similarly, $$\overset{\vee}{T}_y = 2\mu A\left[\frac{\mu}{\lambda+\mu}\left\{\cos(z,\nu)\frac{\partial r^{-1}}{\partial x}-\cos(y,\nu)\frac{\partial r^{-1}}{\partial z}\right\}+3\frac{\partial r}{\partial x}\frac{\partial r}{\partial z}\frac{\partial r^{-1}}{\partial \nu}\right. \quad (3.16)$$

$$\overset{\vee}{T}_z = 2\mu A\frac{\partial r^{-1}}{\partial \nu}\left\{3\left(\frac{\partial r}{\partial z}\right)^2+\frac{\mu}{\lambda+\mu}\right\},$$

where $\hat{\nu}$ is the inward drawn unit normal to a sphericai surface with centre at the origin so that

$$\cos(x,\nu)=-\frac{x}{r},\ \cos(y,\nu)=-\frac{y}{r},\ \cos(z,\nu)=-\frac{z}{r},\frac{\partial}{\partial \nu}=-\frac{\partial}{\partial r}$$

Hence from (3.16) we have

$$\overset{\vee}{T}_x = 2\mu A\left[3.\frac{x}{r}.\frac{z}{r}.\frac{1}{r^2}\right]$$

i.e. $\overset{\vee}{T}_x = \dfrac{6\mu Azx}{r^4}$,

and similarly, $\overset{\vee}{T}_y = \dfrac{6\mu Ayz}{r^4}$, $\overset{\vee}{T}_z = \dfrac{2\mu A}{r^2}\left(3.\dfrac{z^2}{r^2}+\dfrac{\mu}{\lambda+\mu}\right)$. (3.17)

Now let X, Y, Z be the resultant traction on the surface of the spherical cavity. Then, we have

$$X = \iint \overset{\vee}{T}_x \, ds = 6\mu A \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{r\sin\theta . r\sin\phi\cos\theta}{r^4} . r^2 \sin\theta d\theta d\theta d\phi = 0$$

$$Y = \iint \overset{\vee}{T}_y \, ds = 6\mu A \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{r\sin\theta . r\sin\phi\cos\theta}{r^4} . r^2 \sin\theta d\theta d\theta d\phi = 0$$

$$Z = \iint \overset{\vee}{T}_z \, ds = 2\mu A \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{r^2}\left(3.\frac{r^2\cos^2\theta}{r^2} + \frac{\mu}{\lambda+\mu}\right) . r^2 \sin\theta d\theta d\phi$$

$$= 2\mu A . 2\pi \int_{\theta=0}^{\pi} \left(3\cos^2\theta + \frac{\mu}{\lambda+\mu}\right) \sin\theta d\theta$$

$$= 4\pi\mu A\left(3.\frac{2}{3} + \frac{\mu}{\lambda+\mu}.2\right)$$

$$= \frac{8\pi\mu A(\lambda+2\mu)}{\lambda+\mu}$$

If L, M, N are the moments of the resultant forces about the coordinate axes, then

$$L = \iint\left(y\overset{\vee}{T}_z - z\overset{\vee}{T}_z\right) ds = \iint\left[\frac{2\mu A y}{r^3}\left(\frac{3z^2}{r^2} + \frac{\mu}{\lambda+\mu}\right) - z.\frac{6\mu A yz}{r^4}\right] ds$$

$$= \frac{2\mu^2 A}{\lambda+\mu} \iint \frac{y}{r^3} ds = \frac{2\mu^2 A}{\lambda+\mu} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{r\sin\theta\sin\phi}{r^3} . r^2 \sin\theta d\theta d\phi = 0$$

Similarly, $M = \iint\left(z\overset{\vee}{T}_x - x\overset{\vee}{T}_z\right) ds = 0$, $N = \iint\left(x\overset{\vee}{T}_y - y\overset{\vee}{T}_x\right) ds = 0$.

Thus, whatever be the radius of the catity, the system of tractions (3.17) is statically equivalent to a single force of magnitude $\frac{8\pi\mu A(\lambda+2\mu)}{\lambda+\mu}$ applied at the origin along the positive z-axis.

The solutions of equations (3.12) in the form (3.13) are called **first type of simple solutions.**

II. Second Type :

Consider the displacements

$$u = B.\frac{x}{r(z+r)},\ v = B.\frac{y}{r(z+r)},\ w = \frac{B}{r},\ \left(r^2 = x^2 + y^2 + z^2\right)$$

i.e. $u = B.\dfrac{\partial}{\partial x}\log(z+r),\ v = B.\dfrac{\partial}{\partial y}\log(z+r),\ w = B.\dfrac{\partial}{\partial z}\log(z+r)$, (3.18)

being a constant. It may be readily verified that these displacements are the solutions of the equations (3.12) at all points except the origin and the points on the negative z-axis. Now

$$\vartheta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = B\left[\left\{\frac{1}{r(z+r)} - \frac{x}{r^2(z+r)}.\frac{n}{r} - \frac{x}{r(z+r)^2}.\frac{x}{r}\right\}\right.$$

$$\left. + \left\{\frac{1}{r(z+r)} - \frac{y}{r^2(z+r)}.\frac{y}{r} - \frac{y}{r(z+r)^2}.\frac{y}{r}\right\} - \frac{1}{r^2}.\frac{z}{r}\right]$$

=0.

The stress components are

$$\tau_{xx} = \lambda\vartheta + 2\mu\frac{\partial u}{\partial x} = 2\mu B\left[\frac{1}{r(z+r)} - \frac{x^2}{r^2(z+r)} - \frac{x^2}{r^2(z+r)^2}\right]$$

i.e. $\tau_{xx} = 2\mu B\left[\dfrac{y^2+z^2}{r^3(z+r)} - \dfrac{x^2}{r^2(z+r)^2}\right]$

Similarly, $\tau_{yy} = 2\mu B\left[\dfrac{z^2+x^2}{r^3(z+r)} - \dfrac{y^2}{r^2(z+r)^2}\right]$, (3.19)

$$\tau_{zz} = -2\mu B\frac{z}{r^3},$$

$$\tau_{yz} = -2\mu B.\frac{y}{r^3},\ \tau_{zx} = -2\mu B.\frac{x}{r^3},\ \tau_{xy} = -2\mu B.\frac{xy(z+2r)}{r^3(z+r)^2}$$

At the surface of a hemisphere, for which r is constant and z positive.

$$\cos(x,\nu) = -\frac{x}{r},\ \cos(y,\nu) = -\frac{y}{r},\ \cos(z,\nu) = -\frac{z}{r},$$

and the tractions are

$$\overset{\nu}{T}_x = \tau_{xx}\cos(x,\nu) + \tau_{xy}\cos(y,\nu) + \tau_{xz}\cos(z,\nu)$$

$$= 2\mu B\left[\frac{y^2+z^2}{r^3(z+r)} - \frac{x^2}{r^2(z+r)^2}\right]\left(-\frac{x}{r}\right) - 2\mu B.\frac{xy(z+2r)}{r^3(z+r)^2}\left(-\frac{y}{r}\right) - 2\mu B.\frac{x}{r^3}\left(-\frac{z}{r}\right)$$

i.e. $\overset{\nu}{T}_x = \dfrac{2\mu Bx}{r^2(z+r)}$.

Similarly, $\overset{\nu}{T}_y = \dfrac{2\mu By}{r^2(z+r)}, \overset{\nu}{T}_z = \dfrac{2\mu B}{r^2}$

In the above, the unit normal vector $\hat{\upsilon}$ is taken towards the centre.
Let X, Y, Z be resultant traction over the hemispherical surface.
Then

$$X = \iint \overset{\nu}{T}_x\, ds = \frac{2\mu Bx}{r^2(z+r)}ds = 2\mu B\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\frac{r\sin\theta\cos\phi}{r^2(r\cos\theta+r)}.r^2\sin\theta\, d\theta\, d\phi = 0$$

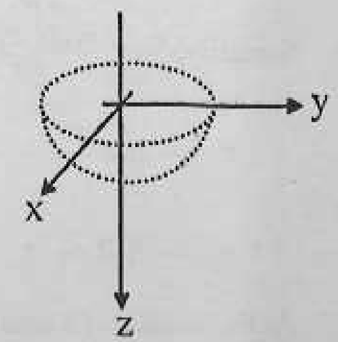
Similarly, $Y = \iint \overset{\nu}{T}_y\, ds = 0$, $Z = \iint \overset{\nu}{T}_z\, ds = 4\pi\mu B$.

It can also be seen that the moments of the resultant forces about the coordinate axes are zero.

The displacement expressed by (3.18) are called **second type of simple solutions.**

III. Superposition of two solutions—Point source of the origin of a semi-infinite medium.

Consider an elastic body subjected to forces applied in the neighbourhood of a single point on the surface. If we suppose that all the linear dimensions of the body are large in comparisor with the area subjected to the load, then the body may be regarded to be bounded by an infinite plane.



Let the force be applied at the origin on the plane $z = 0$ of the semi-infinite medium $z \geq 0$. We exclude the origin by a hemispherical surface as the local effect of the applied force is very great.

Now we have already seen in the first type that the displacement expressed by

$$u = \frac{Azx}{r^3}, v = \frac{Ayz}{r^3}, w = A\left(\frac{z^2}{r^3} + \frac{\lambda+3\mu}{\lambda+\mu}.\frac{1}{r}\right) \tag{3.20}$$

could be maintained in the body by tractions over the plane boudary $z = 0$ given by

$$\tau_{zx} = -\frac{2\mu^2 A}{\lambda+\mu}.\frac{x}{r^3}; \tau_{yz} = -\frac{2\mu^2 A}{\lambda+\mu}.\frac{y}{r^3}, \tau_{zz} = 0$$

and by tractions over the hemispherical boundary given by

$$\overset{\vee}{T}_x = 6\mu A.\frac{zx}{r^4}, \overset{\vee}{T}_y = 6\mu A.\frac{yz}{r^4}, \overset{\vee}{T}_z = \frac{2\mu A}{r^2}\left(3.\frac{z^2}{r^2}+\frac{\mu}{\lambda+\mu}\right),$$

$\hat{\upsilon}$ being the inward drawn normal to the hemispherical surface. The resultant of the latter for the hemispherical surface is a force or magnitude $4\pi\mu A(\lambda+2\mu)/(\lambda+\mu)$ at the origin in the positive direction of the z-axis.

Also from the second type, the displacements given by

$$u = B.\frac{x}{r(z+r)}, v = B.\frac{y}{r(z+r)}, w = \frac{B}{r} \qquad (3.21)$$

could be maintained in the body tractions over the plane boundary $z = 0$ as

$$\tau_{zx} = -2\mu B.\frac{x}{r^3}, \tau_{yz} = -2\mu B.\frac{y}{r^3}, \tau_{zz} = 0$$

and by tractions over the hemispherical surface as

$$\overset{\vee}{T}_x = 2\mu B.\frac{x}{r^2(z+r)}, \overset{\vee}{T}_y = 2\mu B.\frac{y}{r^2(z+r)}, \overset{\vee}{T}_z = \frac{2\mu B}{r^2}.$$

The resultant of these tractions is a force of magnitude $4\pi\mu B$ at the origin in the positive direction of the z-axis.

Let $B = -\frac{A\mu}{\lambda+\mu}$. Then the state of displacement expressed by the sum of the displacements (3.20) and (3.21) will be maintained by forces applied to the hemispherical surface only. If the resultant of these forces is P, then

$$P = \frac{4\pi\mu A(\lambda+2\mu)}{\lambda+\mu} + 4\pi\mu B = \frac{4\pi\mu A(\lambda+2\mu)}{\lambda+\mu} - \frac{4\pi\mu^2 A}{\lambda+\mu} = 4\pi\mu A$$

so that $A = \frac{P}{4\pi\mu}, B = -\frac{P}{4\pi(\lambda+\mu)}$.

Hence the displacements are given by

$$u = \frac{P}{4\pi\mu}.\frac{zx}{r^3} - \frac{P}{4\pi(\lambda+\mu)}.\frac{x}{r(z+r)},$$

$$v = \frac{P}{4\pi\mu} \cdot \frac{yz}{r^3} - \frac{P}{4\pi(\lambda+\mu)} \cdot \frac{y}{r(z+r)} \tag{3.22}$$

$$w = \frac{P}{4\pi\mu} \cdot \frac{z^2}{r^3} + \frac{P(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)} \cdot \frac{1}{r}.$$

3.4 Summary

Some problems of deformation of semi-infinite elastic solid with given displacements or stresses on the plane boundary are considered. As particular cases, the deformation of the body subject to distributed normal pressure on the plane boundary or at a point there has been obtained. Moreover, the types of forces for given displacements in various forms are also considered.

3.5 Exercises

1. Short Answer Type :

(a) State first type (second type) of simple solutions.

(b) State the points of discontinuity in the displacements of first (second type) of simple solutions.

(c) Find out the displacements at interior points of a semi-infinite solid $z \geq 0$ when $u = 0$, $v = 0$, $w = A/(x'^2 + y'^2)$ on the boundary $z = 0$, where A is constant and $(x', y', 0)$ is a point on $z = 0$.

(d) Solve the problem of deformation of a semi-infinite solid $z \geq 0$ with given displacements on the plane boundary.

2. Broad Answer Type :

(a) Find out the displacements and stresses within a semi-infinite solid $z \geq 0$ subject to given surface tractions on the plane boundary $z = 0$.

(b) Solve the problem of deformation of a semi-infinite solid subject to distributed normal pressure on the plane boundary.

(c) Solve the problem of deformation of a semi-infinite solid subject to a normal constant pressure at a point on the plane boundary.

(d) Show that the displacements expressed by

$$u = A.\frac{zx}{r^3},\ v = A.\frac{yz}{r^3},\ w = A\left(\frac{z^2}{r^3}+\frac{\lambda+3\mu}{\lambda+\mu}.\frac{1}{r}\right)$$

can be produced in an elastic body by a single force of magnitude $8\pi\mu(\lambda+2\mu)A/(\lambda+\mu)$ applied at the origin in the direction of z-axis, where A is constant and $r^2 = x^2+y^2+z^2$.

(e) Determine the magnitude of the force which gives rise to the displacements

$$u = B.\frac{x}{r(z+r)},\ v = B.\frac{y}{r(z+r)},\ w = B.\frac{z}{r}$$

or, $u = B\frac{\partial}{\partial x}\log(z+r),\ v = B\frac{\partial}{\partial y}\log(z+r),\ w = B\frac{\partial}{\partial z}\log(z+r),$

where B is constant and $r^2 = x^2+y^2+z^2$, in a hemisphere at the origin along the direction of the positive z-axis, the origin being at the centre of the hemisphere on the plane area.

(f) Find the displacement in a hemisphere if a constant force of magnitude P is applied to the hemispherical surface along the direction of the positive z-axis.

Unit-4 □ Variational Methods

Structure

4.1 Introduction

Variational method is based on the fact that the governing partial differential equation of an elastic problem can be obtained as a direct consequence of minimization of a certain energy experiment. Instead of solving the differential equation we may therefore seek a solution which minimizes the energy experiment and may therefore avoid the mathematical difficulty in obtaining solution of such differential equation. In the development of this method we shall make use of the calculus of variation.

4.2 Euler's Equation

We use the term 'functional' to describe function defined by integrals where argument themselves are

$$I(y)=\int_{x_0}^{x_1} F(x,y,y')dx \tag{4.1}$$

where $F(x,y,y')$ is a known real function of the real arguments $x,y,y'\left(=\frac{dy}{dx}\right)$ and

$y = y(x)$. We consider the functions $y(x) \in C^2(x_0, x_1)$ and assume

$$\left.\begin{aligned} y(x_0) &= y_0, \\ y(x_1) &= y_1, \end{aligned}\right\} \qquad (4.2)$$

y_0 and y_1 being prescribed in advance. The entire set $\{y(x)\}$ of admissible argument $y(x)$ can thus be viewed as a family of smooth curves passing through (x_0, y_0) and (x_1, y_1). We assume further that $F(x, y, y')$ has its first and second order derivatives continuous for all values of y' is some specified region of xy-plane containing the curves $\{y(x)\}$. If $I(\bar{y})$ be definite numerical value of integral (4.1) for the curve $y = \bar{y}(x)$ of the set $\{y(x)\}$. We now proceed to seek the particular curve of the set $\{y(x)\}$ which makes (4.1) a minimum.

Let $y(x)$ minimizes the integral (4.1), then we can represent every function $\bar{y}(x)$ as

$$\bar{y}(x) = y(x) + \varepsilon\eta(x) \qquad (4.3)$$

where ε is a small real parameter and $y(x)$ is determined with $\varepsilon = 0$. Then the variation of $y(x)$ is

$$\delta y = \bar{y}(x) - y(x) = \varepsilon\eta(x)$$

by (4.2). Since every functions in the set $\{y(x)\}$ satisfies the end condition (4.2), so

$$\bar{y}(x_0) = y(x_0), \ \bar{y}(x_1) = y(x_1)$$

and hence $\eta(x_0) = \eta(x_1) = 0$ (4.4)

Since $y(x)$ minimizes (4.1), therefore

$$I(\bar{y}) \geq I(y)$$

i.e. $I(y + \varepsilon\eta) \geq I(y)$ (4.5)

As the L.H.S. of (4.5) is continuously differentiable of ε, so the necessary condition that $y(x)$ minimizes (4.1) is

$$\frac{d}{d\varepsilon} I(y + \varepsilon\eta) \Big|_{\varepsilon=0} = 0 \qquad (4.6)$$

Now,

$$I(y + \varepsilon\eta) = \int_{x_0}^{x_1} F(x, y, +\varepsilon\eta, y' + \varepsilon\eta')\, dx$$

and on differentiating under the integral sign we obtain

$$e\frac{d}{d\varepsilon}I(y+\varepsilon\eta)=\int_{x_0}^{x_1}\left(\eta F_y+\eta' F_{y'}\right)dx \tag{4.7}$$

Integrating by parts the second term on the R.H.S. of (4.7) we get

$$\int_{x_0}^{x_1} F_{y'}\eta' dx = F_{y'}\eta\Big|_{x_0}^{x_1}-\int_{x_0}^{x_1}\eta\frac{dF_{y'}}{dx}dx$$

$$=-\int_{x_0}^{x_1}\eta\frac{dF_{y'}}{dx}dx.$$

Thus (4.7) stands as,

$$\int_{x_0}^{x_1}\left(F_y-\frac{dF_{y'}}{dx}\right)\eta(x)dx=0 \tag{4.8}$$

Since the function $\eta(x)$ is such that its first and second derivatives are continuous and $\eta(x_0)=\eta(x_1)=0$,

$$F_y-\frac{dF_{y'}}{dx}=0 \tag{4.9}$$

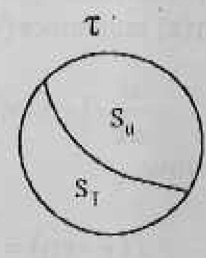
Equation (4.9) is a necessary condition that $y(x)$ makes (4.1) minimum and is called **Euler's equation associated with the variational problem $I(y)$=min.**

4.3 Theorem of Minimum Potential Energy

Theorem -1 : Of all displacements which satisfy the given boundary conditions, those satisfying the equilibrium equations, make the potential energy minimum.

Proof : Let us consider a body τ bounded by a surface S action of specified body forces F_i and surface forces T_i.

The surface S may be divided into two parts. In one part S_T, the surface forces T_i are prescribed while on the remaining part S_u, the displacements are prescribed. In consideration of virtual displacements, δu_i, consistent with constraints imposed on the body, the portion S_u will have no contribution. So in our due calculations without any loss of generality, we denote the surface S, instead of S_T.



Let, u_i be stresses and displacements within the body and the system be given a virtual displacements δu_i consistent with the constraints. Then the virtual work done by the force F_i and $\overset{v}{T}_i$ is

$$\delta U = \int_\tau F_i \delta u_i \, d\tau + \int_s \overset{v}{T}_i \delta u_i \, ds \tag{4.10}$$

where the strain energy U is given by

$$U = \int_\tau W d\tau, \text{ where } W = \tfrac{1}{2}\lambda\vartheta^2 + \mu e_{ij} e_{ij} \tag{4.11}$$

The strain energy U is equal to the work done by the external forces on the body in bringing it from the natural state to the state of equilibrium characterized by the displacements u_i.

Now, let δu_i be the arbitrary variations of the displacements u_i and the volume τ is fixed. Then F_i and $\overset{v}{T}_i$ do not vary so that we can write (4.10) in the form

$$\delta \int_\tau W d\tau = \delta U = \delta\left[\int_\tau F_i u_i \, d\tau + \int_S \overset{v}{T}_i u_i d\tau\right]$$

i.e. $$\delta\left[\int_\tau W d\tau - \int_\tau F_i u_i \, d\tau - \int_S \overset{v}{T}_i u_i d\tau\right] = 0$$

This shows that the expression within the square bracket has a stationary value for admissible variations δu_i of u_i of the equilibrium state.

Defining the potential energy V by the formula

$$V = \int_\tau W d\tau - \int_\tau F_i u_i \, d\tau - \int_S \overset{v}{T}_i u_i d\tau \tag{4.12}$$

we see from the above formula that $\delta V = 0$.

We now show that the increment ΔV of V by replacing the equilibrium displacements u_i by $u_i + \delta u_i$ is positive for all nonvanishing variations δu_i.

Noting that $W = \frac{1}{2}\lambda\vartheta^2 + \mu e_{ij} e_{ij}$, it follows that

$$\Delta W = \left(\tfrac{1}{2}\lambda\vartheta^2 + \mu e_{ij} e_{ij}\right)\Big|_{u+\delta u} - \left(\tfrac{1}{2}\lambda\vartheta^2 + \mu e_{ij} e_{ij}\right)\Big|_{u}$$

where $e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$. Since

$$e_{ij}|_{u+\delta u} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) + \tfrac{1}{2}\left[(\delta u_i)_{,j} + (\delta u_j)_{,i}\right] = e_{ij} + \tfrac{1}{2}(\delta u_i)_{,j} + \tfrac{1}{2}(\delta u_j)_{,i}$$

so that $\vartheta|_{u+\delta u} = e_{ii} + (\delta u_i)_{,i} = \vartheta + (\delta u_i)_{,i}$ and therefore,

$$\Delta W = \tfrac{1}{2}\lambda\left[\vartheta + (\delta u_i)_{,i}\right]\left[\vartheta + (\delta u_j)_{,j}\right]$$

$$+\mu\left[e_{ij} + \tfrac{1}{2}(\delta u_i)_{,j}\tfrac{1}{2}(\delta u_j)_{,i}\right]\left[e_{ij} + \tfrac{1}{2}(\delta u_i)_{,j} + \tfrac{1}{2}(\delta u_j)_{,i}\right]$$

$$-\tfrac{1}{2}\lambda\vartheta^2 - \mu e_{ij}e_{ij}$$

$$= \lambda\vartheta(\delta u_i)_{,i} + 2\mu e_{ij}(\delta u_i)_{,j} + P \tag{4.13}$$

where $P = \tfrac{1}{2}\lambda\left[(\delta u_i)_{,i}\right]^2 + \dfrac{\mu}{4}\left[(\delta u_i)_{,j} + (\delta u_j)_{,i}\right]^2 \geq 0$.

It is to be noted that $P = 0$ only when $\delta e_{ij} = \tfrac{1}{2}\left[(\delta u_i)_{,j} + (\delta u_j)_{,i}\right] = 0$.

Now, since $\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}$, we can rewrite (4.13) as

$$\Delta W = \lambda\vartheta\delta_{ij}(\delta u_i)_{,j} + 2\mu e_{ij}(\delta u_i)_{,j} + P$$

$$= (\lambda\vartheta\delta_{ij} + 2\mu e_{ij})(\delta u_i)_{,j} + P$$

$$= \tau_{ij}(\delta u_i)_{,j} + P. \tag{4.14}$$

Thus the increment ΔU in the strain energy U is given by the use of (4.11) and (4.14) as

$$\Delta U = \int_\tau \Delta W d\tau = \int_\tau \tau_{ij}(\delta u_i)_{,j}\,d\tau + \int_\tau P d\tau$$

$$= \int_\tau (\tau_{ij}\delta u_i)_{,j}\,d\tau - \int_\tau \tau_{ij,j}\delta u_i\,d\tau + \int_\tau P d\tau$$

$$= \int_S \tau_{ij}\nu_j\delta u_i\,ds - \int_\tau \tau_{ij,j}\delta u_i\,d\tau + \int_\tau P d\tau$$

$$= \int_S \overset{\nu}{\tau}_i\,\delta u_i\,ds - \int_\tau \tau_{ij,j}du_i\,d\tau + Q \tag{4.15}$$

where $Q \equiv \int_\tau P d\tau \geq 0$. Now, if the body is in equilibrium, then

$$\int_\tau \tau_{ij,j} = -F_i \text{ in } \tau \text{ and } \tau_{ij}\nu_j = \overset{u}{T}_i \text{ on } S.$$

Hence from (4.15), we have

$$\Delta U = \int_S \overset{v}{T}_i \delta u_i ds + \int_\tau F_i \delta u_i d\tau + Q$$

so that by the definition of potential energy given in (4.12), we get

$$\Delta V = \Delta U - \int_S \overset{v}{T}_i \delta u_i ds - \int_\tau F_i \delta u_i d\tau = Q \qquad (4.16)$$

Hence the theorem.

4.4 Theorem of Minimum Complementary Energy

Theorem - 2 : The complementary energy attains its absolute minimum for the equilibrium state of the stress tensor τ_{ij} and varied states of stress satisfy the conditions

$$\left(\delta\tau_{ij}\right)_{,j} = 0 \text{ in } \tau$$

$$\left(\delta\tau_{ij}\right)\nu_j = 0 \text{ on } S_T \qquad (4.17)$$

$\delta\tau_{ij}$ we arbitrary on S_u.

Proof : Let a body τ be in equilibrium under the action of body forces F_i and surface forces $\overset{v}{T}_i \cdot \overset{v}{T}_i$ is assigned over the portion S_T of the surface S and on the remaining portion S_u, the displacement u_i are prescribed.

Since τ_{ij} are the solutions of the equilibrium problem, therefore τ_{ij} satisfy the relations.

(i) $\tau_{ij,j} + F_i = 0$ in τ (4.18)

(ii) $B\left(\tau_{ij}\right) = 0$ where $B\left(\tau_{ij}\right) = 0$ implies that τ_{ij} satisfies Beltrami-Michell Compatibility equations. (4.19)

(iii) $\tau_{ij}\nu_j = T_i$ on S_T, ν_j in the normal measure positive inward. (4.20)

Corresponding to these τ_{ij}, the strain energy is

$$U\left(\tau_{ij}\right) = \int_\tau W\left(\tau_{ij}\right) d\tau \qquad (4.21)$$

Let us now consider virtual variation $\tau_{ij} + \delta\tau_{ij}$ of the stress τ_{ij} where $\delta\tau_{ij}$ satisfies the given condition (4.17). Since

$\tau_{ij} + \delta\tau_{ij}$ are not associated with the equilibrium state of the body, so

$$B\left(\tau_{ij} + \delta\tau_{ij}\right) \neq 0$$

Since B is linear, we have

$$B\left(\tau_{ij}\right) + B\left(\delta\tau_{ij}\right) \neq 0$$

$$B\left(\delta\tau_{ij}\right) \neq 0.$$

Now,

$$U' = U\left(\tau_{ij} + \delta\tau_{ij}\right)$$

$$= \int_{\tau} W\left(\tau_{ij} + \delta\tau_{ij}\right) d\tau$$

$$= \int_{\tau} \left(W\left(\tau_{ij}\right) + W\left(\delta\tau_{ij}\right) + \delta W\right) d\tau$$

$$= \int_{\tau} W\left(\tau_{ij}\right) d\tau + \int_{\tau} W\left(\delta\tau_{ij}\right) d\tau + \int_{\tau} \frac{\partial W}{\partial \tau_{ij}} \delta\tau_{ij}\, d\tau$$

therefore

$$U' - U = \int_{\tau} W\left(\delta\tau_{ij}\right) d\tau + \int_{\tau} e_{ij} \delta\tau_{ij}\, d\tau$$

$$= Q + \int_{\tau} u_{i,j}\, \delta\tau_{ij}\, d\tau$$

$$= Q + \int_{\tau} (u_i \delta\tau_{ij})_{,j}\, d\tau - \int \delta\tau_{ij,j} u_i\, d\tau$$

$$= Q + \int_{\tau} (u_i \delta\tau_{ij})_{,j}\, d\tau, \quad \text{since } \delta\tau_{ij,j} = 0 \text{ on } \tau \text{ (using the cond. of (4.17))}$$

$$= Q + \int_{S} u_i\, (\delta\tau_{ij})\, \nu_j\, ds$$

$$= Q + \int_{S} \delta \overset{\nu}{T}_i\, u_i\, d$$

$$\therefore \delta U = Q + \int_S \delta \overset{\vee}{T}_i u_i \, ds$$

or, $$\delta\left(U - \int_{S_u} \overset{\vee}{T}_i u_i \, ds\right) = Q \geq 0$$ since W is + ve definite (4.22)

$U - \int_{S_u} T_i u_i ds$ is defined as **Complementary energy.**

Hence the theorem follows.

4.5 Reciprocal Theorem of Bethi and Rayleigh

Theorem : If an elastic body is subjected to two systems of body and surface forces then the work done by the first system F_i, T_i in acting through displacements u_i' due to the second system of forces is equal to the work done by the second system F_i', T_i' in acting through the displacement u_i due to the first system of forces.

Proof : Consider two states of an elastic elastic body–one with displacement u_i, body forces F_i and surface forces $\overset{\vee}{T}_i$ and the other with displacement u_i', body forces F_i' and surface forces $\overset{\vee}{T}_i'$.

Now the work done W_{12} by the forces F_i, and $\overset{\vee}{T}_i$ of the first state in acting through displacements u_i' of the second state is given by

$$W_{12} = \int_\tau F_i u_i' \, d\tau + \int_S \overset{\vee}{T}_i u_i' ds$$

$$= -\int \tau_{ij,j} u_i' d\tau + \int \tau_{ij} \nu_j u_i' ds$$

since $\tau_{ij,j} + F_i = 0$ in τ (from equations of equlibrium and $\tau_{ij}\nu_j = \overset{\vee}{T}_i$ on S.

So $$W_{12} = -\int \tau_{ij,j} u_i' d\tau + \int_\tau \left(\tau_{ij} u_i'\right)_{,j} d\tau$$ (by divergence theorem)

$$= -\int_\tau \tau_{ij,j} u_i' d\tau + \int_\tau \left(\tau_{ij,j} u_i' + \tau_{ij} u_{i,j}'\right) d\tau$$

$$= \int_\tau \tau_{ij}\, u'_{i,j}\, d\tau$$

$$= \int_\tau \left[\lambda\vartheta\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)\right] u'_{i,j}\, d\tau, \quad \left[\vartheta = u_{i,i}\right]$$

$$= \int_\tau \left(\lambda\vartheta' + \mu u_{i,j}\, u'_{i,j} + \mu u_{j,i} u'_{i,j}\right) d\tau, \quad \left[\vartheta' = u'_{i,i}\right] \qquad (4.23)$$

Proceeding similarly, the workdone by forces F_i' and T_i' in acting through the displacement u_i is

$$W_{21} = \int_\tau \left(\lambda\vartheta\vartheta' + \mu u'_{i,j}\, u_{i,j} + \mu u'_{j,i}\, u_{i,j}\right) d\tau \qquad (4.24)$$

Compairing W_{12} and W_{21} we find that except for the last term in the integrand the two expressions are the same. Now we observe that

$\mu u_{j,i}\, u'_{i,j} = \mu u_{i,j}\, u'_{j,i} = \mu u'_{j,i}\, u_{i,j}$, (interchanging dummy suffixes *i* and *j*)

so that from (4.23) and (4.24) we find that

$$W_{12} = W_{21}$$

i.e. $$\int_S T_i u_i'\, ds + \int_\tau F_i u_i'\, d\tau = \int_S T_i' u_i\, ds + \int_\tau F_i' u_i\, d\tau \qquad (4.25)$$

Hence the theorem.

4:6 Examples

(1) Deflection of an elastic string

Let a stretched string of lenght l with its ends fixed at (0,0) and (l,0) be deflected by a distributed transverse load $f(x)$ per unit length of the string. Also we suppose that the transverse deflection $y(x)$ is small and that the change in the stretching force T produced by the deflection be negligible.

The potential energy V is given by

$$V = U - \int_0^l f(x) y\, dx \text{ (using 4.14)}$$

where U is the strain energy and is equal to the product of the tensile force T and the total stretch e of the string.

Now

$$e = \text{ total stretch} = \int_0^l (ds - dx)$$

$$= \int_0^{\ell}\left(\sqrt{1+(y')^2}-1\right)dx.$$

But for linear theory, $y'^2 << 1$ and so we can write

$$e = \int_0^{\ell} \tfrac{1}{2}(y')^2\, dx.$$

Consequently,

$$U = \frac{T}{2}\int_0^{\ell}(y')^2\, dx$$

and hence

$$V = \int_0^{\ell}\left(\frac{T}{2}(y')^2 - f(x)y\right)dx$$

Then from (4.9) the appropriate Euler's equation is

$$Ty'' + f(x) = 0$$

This is the equation for the transverse deflection of the string under the load $f(x)$.

(2) Deflection of the central line of a beam :

Let the cross-section of the beam is constant and the x-axis lies along the axis of the beam. It is bent by transverse loading $p = f(x)$ estimated per unit length of the beam. We assume that shearing stresses are negligible in comparision with the tensible stress.

$$\tau_{xx} = \frac{M_y}{I}$$

The strain e_{xx} is then given by

$$e_{xx} = \frac{\tau_{xx}}{E} = \frac{M_y}{EI}$$

The strain energy density is

$$W = \tfrac{1}{2}\tau_{xx}e_{xx} = \frac{M^2y^2}{2EI^2}$$

The strain energy per unit length of the beam is found by integrating over the cross-section (area A) of the beam, and we get

$$\int_A W d\sigma = \frac{M^2}{2EI^2}\int_A y^2 d\sigma = \frac{M^2}{2EI}$$

By Bernoulli Euler law we have,

$$M = EIy'' ;$$

thus

$$\int_A W d\sigma = \frac{EI}{2}(y'')^2$$

The total strain energy U is obtioned by integrating this expression over the length of the beam, and we find

$$U = \int_0^l \tfrac{1}{2} EI(y'')^2 \, dx$$

Now we suppose that the ends of the beam are clamped, hinged, or free so the supporting forces do not work and hence contribute nothing to potential energy V.

If we neglect the weight of the beam, the only externel load is $p = f(x)$ and so we get

$$V = \int_0^l \tfrac{1}{2} EI(y'')^2 \, dx - \int_0^l f(x) y \, dx.$$

$$= \int_0^l \left(\tfrac{1}{2} EI(y'')^2 - f(x) y \right) dx.$$

The Euler equation is

$$\frac{d^2}{dx^2}(EI\, y'') - f(x) = 0$$

4.7 Summary

In this unit the Euler's equation, the theorem of minimum potential energy, theorem of minimum of complementary energy, Reciprocal theorem and some related problems are studied.

4.8 Exercises

I. Short Answer Type :

(a) Based on which principle the variational method is applicable?

(b) Write down the Euler's equation.

(c) State the theorem of minimum potential energy.

2. ***Broad Answer Type :***

(a) Derive Euler's equation using variational method.

(b) Prove that of all displacements satisfying the given boundary conditions those which satisfy the equilibrium equations make the potential energy minimum.

(c) State and prove the reciprocal theorem of Betti and Rayleigh.

(d) In the reciprocal theorem, take $F_i' = 0$, $\tau_{ij}' = \delta_{ij}$, show that $T_i = \tau_{ij}\, \upsilon_j = \upsilon_i$,

$\Theta' = \tau_{ii}' = 3$, $e_{ij}' = \dfrac{1-2\sigma}{E}\delta_{ij}$ and $u_i' = \dfrac{1-2\sigma}{E}x_i$. Insert these expressions in (4.25) and derive the following expression for the change in volume ΔV_0 in an elastic body under the action of surface forces T_i and body forces t_i :

$$\Delta V_0 \equiv \int_\tau \theta d\tau = \frac{1-2\sigma}{E}\left(\int_S T_i x_i ds + \int_\tau F_i x_i d\tau\right)$$

(e) Obtain the equation for the transverse deflection of the string under the load $f(x)$ per unit length using principle of minimum potential energy.

(f) Show that the differential equation

$\dfrac{d^2}{dx^2}(EIy'') - f(x) = 0$ represents the deflection of the central line of the beam.

Unit-5 □ Elastic Waves

Structure

5.1 Introduction

In an elastic medium, the action of a sudden disturbance is transmitted at once to other part of the body. At the beginning, the remote parts of the medium are not disturbed and the deformations produced at a point are propagated through the medium in the form of waves, known as **elastic waves**. In this unit we shall discuss various types of elastic waves and their characteristics.

5.2 Body Waves

1. Waves of dilatation and waves of distortion

The equation of motion for a homogeneous, isotropic and perfectly elastic medium in the absence of body forces, in vector form is

$$(\lambda+\mu)\ \text{grad div}\ \vec{q}+\mu\nabla^2\vec{q}=\rho\frac{\partial^2\vec{q}}{\partial t^2}, \tag{5.1}$$

where $\vec{q}=(u,v,w)$ in the displacement vector measured relative to the reference state.

Taking divergence on both sides of (5.1) we have (noting that $\vartheta = div\vec{q}$)

$$(\lambda+\mu)\ \nabla^2(\vartheta)+\mu\nabla^2(\theta)=\rho\frac{\partial^2(\vartheta)}{dt^2}$$

i.e. $\nabla^2(\theta) = \frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}(\vartheta)$ (5.2)

where $\alpha = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}$ (5.3)

Again taking the curl on both sides of (5.1) and noting that curl grad (a scalar) $= \bar{O}$ and that $\bar{w} = \frac{1}{2}$ curl $\bar{q}$ is the rotation vector, we get

$$\mu\nabla^2\bar{w} = \rho\frac{\partial^2(\bar{w})}{\partial t^2}$$

i.e. $\nabla^2\bar{w} = \frac{1}{\beta^2}\frac{\partial^2(\bar{w})}{\partial t^2}$ (5.4)

where $\beta = \left(\mu/\rho\right)^{1/2}$ (5.5)

Equation (5.2) and (5.4) are both wave equations. From (5.2), it follows that a dilatational disturbance can be transmitted through the medium with a velocity α, given by (5.3), while from (5.4) it follows that a rotational disturbance can be propageted through the medium with a velocity β given by (5.5). The possibility of propagation of dilational wave and distortions wave through the interior of a homogeneous, isotropic elastic solid was first established theoretically by Poisson in 1835.

The dialatational and distortional (rotational) waves propagating through the interior of the elastic body are known as P-wave (stands for primary or 'push' wave) and the S-wave (representing the secondary or 'shake' wave).

II. Plane Waves :

If a disturbance is produced at a point in an elastic medium then the waves radiate from this point in all directions. At a great distance from the centre of disturbance, however, such waves can be considered as **plane wave**. The most general form of a plane wave type disturbance propagating through the medium, can be represented as

$(u, v, w) = (U, V, W)(lx + my + nz - ct),$ (5.6)

where l, m, n are the d.c's of the direction of propagation of plane waves (i.e. normal to the wave front) and c is velocity of propagation. Now from (5.6)

$$\frac{\partial u}{\partial x} = lU', \ \frac{\partial u}{\partial y} = mU', \frac{\partial u}{\partial z} = nU', ...$$

and $\frac{\partial u}{\partial t} = -CU', \frac{\partial v}{\partial t} = -CV', \frac{\partial w}{\partial t} = -CW'$

where primes denote differentiation w.r.t. the arguments.
Then the equation of motion in (5.1) reduces to

$$(\lambda+\mu)\frac{\partial\vartheta}{\partial x} + \mu\nabla^2 u = \rho\ddot{u}$$

i.e $(\lambda+\mu)l(lU''+mV''+nW'')+\mu(l^2+m^2+n^2)U''=\rho c^2 U''$.

or, $\{(\mu-\rho c^2)+(\lambda+\mu)l^2\}U''+(\lambda+\mu)lmV''+(\lambda+\mu)lnW''=0$ (5.7)

$$(\because l^2+m^2+n^2=1)$$

Similarly, we have, from the other two equations of (5.1)

$$(\lambda+\mu)mlU''+((\mu-\rho c^2)+(\lambda+\mu)m^2)V''+(\lambda+\mu)mnW''=0 \quad (5.8)$$

and $(\lambda+\mu)nlU''+(\lambda+\mu)nmV''+((\mu-\rho c^2)+(\lambda+\mu)n^2)W''=0$ (5.9)

Eliminating U'', V'', W'' from (5.7) – (5.9)
and writing $(\lambda+\mu)=A$ and $\mu-\rho c^2=B$, we get

$$\begin{vmatrix} l^2A+B & lmA & lnA \\ mlA & m^2A+B & mnA \\ nlA & mnA & n^2A+B \end{vmatrix} = 0 \quad (5.10)$$

Now applying the operations lc_2-mc_1, lc_3-nc_1, we obtain

$$\begin{vmatrix} (l^2A+B) & -mB & -nB \\ lmA & lB & 0 \\ lnA & 0 & lB \end{vmatrix} = 0 \text{ i.e. } B^2 \begin{vmatrix} l^2A+B & -m & -n \\ lmA & l & 0 \\ lnA & 0 & l \end{vmatrix} = 0.$$

i.e. $l^2B^2\left[A(l^2+m^2+n^2)+B\right]=0$, i.e. $l^2B^2(A+B)=0$ (5.11)

Therefore, either $A+B=0$ or $B=0$, so that either

$$c^2 = \frac{\lambda+2\mu}{\rho} = \alpha^2 \text{ or, } c^2 = \frac{\mu}{\rho} = \beta^2. \quad (5.12)$$

Thus, a plane wave type disturbance can propagate through the material of a perfect elastic, homogeneous, isotropic medium, with only two possible distinct velocity—those of the P-and S-waves.

5.3 Surface Waves

In an unbounded elastic matirial, energy is transmitted by waves of dilatation, often referred to as *primary or compressed waves,* which propagate with velocity $\alpha=\left(\frac{\lambda+2\mu}{\rho}\right)^{½}$, or by waves of distortion, often referred to as *secondary or shear waves* propagating with velocity $\beta=\left(\frac{\mu}{\rho}\right)^{½}$. When there is a boundary, as in a half-space problem, a third type of wave may exist whose effect are confined closely to the surface. These waves were first investigated by Lord Rayleigh, who showed that their effect decreases rapidly with depth and their velocity of propagation is smaller than that of body waves. Here we consider two types of surface waves, namely Rayleigh wave and Love wave.

I. Rayleigh Wave : On the surface of an elastic solid it is possible to have waves of the type which are propagated over the surface and which penetrate but a little distance into the interior of the body. These are known as **Rayleigh waves**. We assume that the body is bounded by the plane z=0 and take the positive sense of the z-axis in the direction towards the interior of the body, the positive direction of x-axis in the direction of wave propagation, there being no displacements in the surface perpendicular the direction of propagation. The displacement components at any point are (u, o, w). The equations of motion are,

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\vartheta}{\partial x} &+\mu\nabla^2u=\rho\ddot{u}\\ (\lambda+\mu)\frac{\partial\vartheta}{\partial y} &=0\\ (\lambda+\mu)\frac{\partial\vartheta}{\partial z} &+\mu\nabla^2w=\rho\ddot{w}\end{aligned}\right\}\qquad(5\text{-}13)$$

Case - I : Let $\vartheta\neq0$, then the rotation is zero and

$$\left.\begin{aligned} \text{therefore } \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} = 0 \text{ since } v = 0 \\ \frac{\partial w}{\partial y} &= \frac{\partial v}{\partial z} = 0 \text{ since } v = 0 \end{aligned}\right\} \tag{5.14}$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \tag{5.15}$$

Conditions (5.14) show that u and w are independent of y and condition (5.15) enables us to write

$$u = \frac{\partial \phi}{\partial x}, \quad w = \frac{\partial \phi}{\partial z}$$

where ϕ is a scalar function

so that $\vartheta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \nabla^2 \phi$

Noting that

$$\frac{\partial \vartheta}{\partial x} = \nabla^2 \frac{\partial \phi}{\partial x} = \nabla^2 u$$

and $\frac{\partial \vartheta}{\partial z} = \nabla^2 w$,

the equations of motion (5.13) reduce to

$$\begin{aligned} (\lambda + 2\mu)\nabla^2 u &= \rho \ddot{u} \\ (\lambda + 2\mu)\nabla^2 w &= \rho \ddot{w} \end{aligned} \tag{5.16}$$

For the solution of (5.16) we assume values of the displacement consistent with the relations (5.14) and (5.15), as

$$\begin{aligned} u &= s e^{-rz} \sin(sx - pt) \\ w &= r e^{-rz} \cos(sx - pt), \end{aligned} \tag{5.17}$$

in which, s, p, r are constants. The amplitude of the wave dimnishes rapidly with increase of depth, r being positive. The velocity of propagation of wave is

$$c = \frac{p}{s} \tag{5.18}$$

Using (5.17) in (5.16) we have

$$r^2 = s^2 - h^2 \tag{5.19}$$

where $h^2 = \dfrac{p^2}{\alpha^2}$ (5.20)

Case-II

Let $\vartheta = 0$. Then we have from (5.13)

$$\begin{aligned} \mu\nabla^2 u &= \rho\ddot{u} \\ \mu\nabla^2 w &= \rho\ddot{w} \end{aligned} \tag{5.21}$$

To satisfy these equations we have

$$\begin{aligned} u &= Abe^{-bz}\sin(sx-pt) \\ w &= Ase^{-bz}\cos(sx-pt) \end{aligned} \tag{5.22}$$

subject to $\dfrac{\partial u}{\partial x}+\dfrac{\partial w}{\partial z}=0$

putting (5.22) in (5.21). We get

$$b^2 = s^2 - k^2 \tag{5.23}$$

where $k^2 = \dfrac{p^2}{\beta^2}$ (5.24)

Thus the solution of (5.13) can be written as a linear combination of (5.17) and (5.22) in the form

$$\begin{aligned} u &= \left(se^{-rz} + Abe^{-bz}\right)\sin(sx-pt) \\ w &= \left(re^{-rz} + Ase^{-bz}\right)\cos(sx-pt) \end{aligned} \tag{5.25}$$

The boundary conditions are

$\tau_{xz} = \tau_{yz} = \tau_{zz} = 0$ when $z = 0$ i.e., on the surface.

Now,

$$\left.\begin{aligned} \tau_{xz} &= \mu e_{xz} = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0 \\ \tau_{yz} &= \mu e_{yz} = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial y}\right)=0 \\ \tau_{zz} &= \lambda\vartheta + 2\mu e_{zz} = \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}=0 \end{aligned}\right\} \text{ when } z = 0.$$

The first and the 3rd conditions give

$$2rs + A\left(2p^2 - k^2\right) = 0 \tag{5.26}$$

$$\left(2s^2 - k^2\right) + 2Abs = 0 \tag{5.27}$$

Eliminating A for (5.26) and (5.27), we get

$$\left(\frac{k^2}{s^2} - 2\right)^4 = 16\left(1 - \frac{k^2}{s^2}\right)\left(1 - \frac{h^2}{s^2}\right)$$

using (5.18), (5.19), (5.23) and (5.24) we have

$$\left(2 - \frac{c^2}{\beta^2}\right)^4 = 16\left(1 - \frac{c^2}{\alpha^2}\right)\left(1 - \frac{c^2}{\beta^2}\right) \tag{5.28}$$

This is known s the **frequency equation for Rayleigh wave.**

Substituting $\xi = \frac{c^2}{\beta^2}$ and $q = \left(\frac{\beta}{\alpha}\right)^2$ in (5.28) it follows that (5.29)

$$(2 - \xi)^4 = 16(1 - \xi q)(1 - \xi)$$

Or, $\xi\left(\xi^3 - 8\xi^2 + 8(3 - 2q)\xi + 16(q - 1)\right) = 0$ (5.30)

Discarding the solution $\xi = 0$, (as then c = 0, i.e., the velocity is zero which is impossible) we obtain a cubic equation for ξ as

$$f(\xi) = \xi^3 - 8\xi^2 + 8(3 - 2q)\xi + 16(q - 1) = 0 \tag{5.31}$$

$$f(0) = 16(q - 1) = -ve \text{ if } q < 1$$

$$f(1) = 1 = +ve.$$

Hence there must be at least one real between 0 and 1. This proves the existence of a real value of c and hence existence of Rayleigh wave.

A special case is that when the Lamé constants are equal, ie. $\lambda = \mu$ and then the Possion ratio $\sigma = \frac{1}{4}$ and $q = \frac{1}{3}$. The equation (5.31), therefore, reduces to

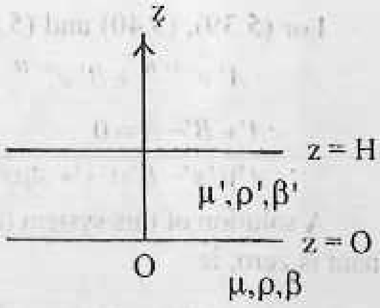
$$\xi^3 - 8\xi^2 + \frac{56}{3}\xi - \frac{32}{3} = 0 \tag{5.32}$$

The roots of this equation are $4, 2 + \frac{2}{\sqrt{3}}, 2 - \frac{3}{\sqrt{3}}$. The first two $(c > \beta)$ corresponds to the case of wave reflection for which $c > \alpha > \beta$.

The value $\frac{c^2}{\beta^2} = 2 - \frac{3}{\sqrt{3}}$ gives $c = 0.9194\beta(<\beta)$, (5.33)

which gives the **velocity of propagation of Rayleigh wave.**

II. Love Waves : The characteristics of Rayleigh waves on the free surface of a half space do not agree with those of the surface waves observed on the earth. Love in 1911 found that, in an elastic layer over a half space, dispersed surface waves are produced with transverse component of displacement v, which propagate in the direction x with velocity c, in a medium of different elastic properties consisting of a layer of thickness H, density ρ' and shear velocity β', over a half space of density ρ and velocity β. We can write the displacement (v, v'), which satisfy the wave equations.



$$\nabla^2 v = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} \text{ and } \nabla^2 v' = \frac{1}{\beta'^2}\frac{\partial^2 v'}{\partial t^2} \quad (5.34)$$

The wave propagated parallel to the axis of x in both the medium, (u, u') and (w, w') are zero where v and v' are given by the relations in the form.

$$v = h(z)e^{ik(x-ct)} \quad (5.35)$$

and $v' = h_1(z)e^{ik(x-ct)}$

sbubstituting (5.35) in (5.34) we have

$$v = Ae^{ik(-sz+x-ct)} + Be^{+ik(sz+x-ct)} \quad (5.36)$$

and $v' = A'e^{ik(-s'z+x-ct)} + B'e^{ik(s'z+x-ct)}$

where $s = \left(\frac{c^2}{\beta^2} - 1\right)^{1/2}$ and $s' = \left(\frac{c^2}{\beta'^2} - 1\right)^{1/2}$

Since the solution (5.36) must be bounded at infinity we most put B=0 and then we have

$$v = Ae^{ik(-sz+x-ct)} \quad (5.37)$$

Again, the amplitude decreases with depth so that

$$c < \beta \quad (5.38)$$

The boundary conditions at the free surface, associated with v', is

$\tau_{yz} = 0$ at $z = H$ (5.39)

and at $z = 0$ i.e., at the interface

$v = v'$ at $z = 0$ (5.40)

and $\tau_{yz} = \tau'_{yz}$ at $z = 0$ (5.41)

For (5.39), (5.40) and (5.41) we get by using (5.36) and (5.37)

$$-A' e^{-iks'H} + B' e^{iks'H} = 0$$

$$A' + B' - A = 0$$

$$A'\mu's' - B'\mu's' + A\mu s = 0$$

A solution of this system (apart from trivial are $A' = B' = A = 0$) requires that determinant is zero, ie

$$\begin{vmatrix} -e^{-iks'H} & +e^{iks'H} & 0 \\ 1 & 1 & 1 \\ -\mu's' & +\mu's' & \mu s \end{vmatrix} = 0$$

or, $\dfrac{\mu s}{\mu' s'} = i \tan(ks'H)$

or, $\mu\, is + \mu' s' \tan(ks'H) = 0$ (5.42)

This is the frequency equation for Love wave for the possibility of propagation of Love wave (5.42) must be satisfied by a value of c such that $\beta' < c < \beta$ which imples that $\beta' < \beta$. [s is $+ve$ imaginary $\therefore$ μ is a real $-ve$ no. so, $\mu' s' \tan(ks'H)$ should be real $+ve$. This requiers that s' is real $+ve$. Hence we must have $(c > \beta')$. The Love wave is of dispersive nature since for (5.42) we see that the velocity c depends on the value of the wave length i.e., $\lambda = \dfrac{2\pi}{k}$.

5.4 Summary

In this unit body waves, surface waves, Rayligh waves and Love waves are discussed.

5.5 Excercises

I. Short Answer Type :

(a) What are P-wave and S-wave?

(b) Define plane wave.

(c) Is Love wave despersive in nature? Verify.

(d) What is the velocity of propagation of Rayleigh wave when the Lame's constants are equal?

2. *Broad Answer Type :*

(a) What are Love waves? Discuss the propagation of Love type wave in a uniform elastic surface layer.

(b) Show that in absence of body forces the dilatational waves and wave of distortion propagate with velocity $\left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}$ and $\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$ respectively.

(c) Show that the plane wave propagate through the elastic material having the velocities of P-wave and S-wave respectively.

(d) What are Rayleigh waves? Show that the velocity C with which Rayleigh waves propagate satisfies the equation

$$\left(2-\frac{c^2}{\beta^2}\right)^2-4\left(1-\frac{c^2}{\alpha^2}\right)^{\frac{1}{2}}\left(1-\frac{c^2}{\beta^2}\right)^{\frac{1}{2}}=0,$$

where the constants α and β are to be defined by you.

Unit-6 □ Transverse Vibration of Thin Elastic Plates

(In this Unit we shall use unabridgod notations in place of tensor notations)

Structure

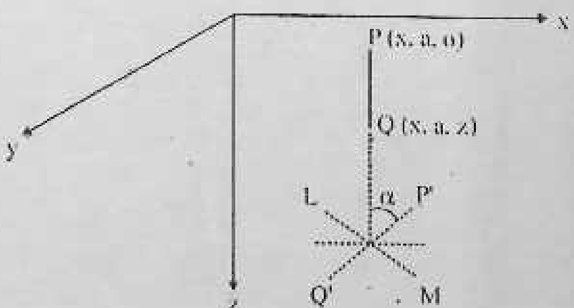
6.1 Basic Preliminaries

For small deflections of thin elastic plates, the following assumptions are made :

(a) The normals to the middle plane before bending are deformed into normals of the middle surface after bending.

(b) The normal component τ_{zz} across each thin layer parallel to the xy-plane is small in comparison to other components of stress and can be neglected.

(c) The slope of the deflected middle surface, called the neutral surface, of the plate in any direction is small.



Also, the plate being thin, its thickness is small compared to its other dimensions. Moreover, assumptions (a) and (b) lead to

$$\tau_{zx} = \tau_{yz} = \tau_{zz} = 0$$

so that each thin layer of the plate parallel to the plane $z = 0$ is in a state of plane stress.

Consider a point $P(x, a, o)$ on the middle plane of a section $y = a$ (constant) and another point Q (x, a, z) downwards of this section so that PQ is parallel to z-axis and perpendicular to x-axis. Suppose P' and Q' are the new positions of P and Q respectively after deformation. Then $PP'(=w)$ is the deflection and $P'Q'$, under assumption (a), remains normal to the deflected curve LM so that the displacement of the point Q parallel to the x-axis is $u = -z\alpha$. But, according to assumption (c), $\alpha \approx \tan\alpha = \frac{\partial w}{\partial x}$. Hence

$$u = -z\frac{\partial w}{\partial x}. \qquad (6.1a).$$

Similarly, the displacement of Q parallel to the g-axis is

$$v = -z\frac{\partial w}{\partial g} \qquad (6.1b)$$

Thus the strain components are

$$e_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2},\ e_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2},\ e_{xy} = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -z\frac{\partial^2 w}{\partial x \partial y}. \qquad (6.2)$$

The stress-strain relations are

$$\tau_{xx} = \frac{E}{1-\sigma^2}\left(e_{xx} + \sigma.e_{yy}\right) = -\frac{Ez}{1-\sigma^2}\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right),$$

$$\tau_{yy} = \frac{E}{1-\sigma^2}\left(e_{yy} + \sigma.e_{xx}\right) = -\frac{Ez}{1-\sigma^2}\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right),$$

$$\tau_{xy} = \frac{E}{1+\sigma}e_{xy} = -\frac{Ez}{1+\sigma}\frac{\partial^2 w}{\partial x \partial y}, \qquad (6.3)$$

where E is the Young's modulus and σ is the Poisson's ratio.

Now let h be the thickness of the plate and M_x and M_y are the bending moments per unit length of sections of the plate perpendicular to the x and y axes respectively. Then

$$M_x = \int_{-h/2}^{h/2} z\tau_{xx}dz,\ M_y = \int_{-h/2}^{h/2} z\tau_{yy}dz. \qquad (6.4)$$

Also, if M_{xy} and M_{yx} be the twisting moments per unit length of section of the plate perpendicular to the x-and y-axis respectively, then

$$M_{xy} = \int_{-h/2}^{h/2} z\tau_{xy}\,dz = \int_{-h/2}^{h/2} z\tau_{yx}\,dz = M_{yx} \qquad (6.5)$$

Substitutng (6.3) into (6.4) and (6.5) and performing the integrations (noting that w is independent of z), we get

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right),\ M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right),$$

$$M_{xy} = M_{yx} = -(1-\sigma)D\frac{\partial^2 w}{\partial x\partial y} \qquad (6.6)$$

where $$D = \frac{Eh^3}{12(1-\sigma^2)} \qquad (6.7)$$

is called the **flexural rigidity** or **bending rigidity** of the plate.

6.2 Differential Equation of Transverse Vibration of Thin Plate

Let Q_x and Q_y be the normal shear forces the plate sustains per unit length parallel to y- and x-axis respectively. Then

$$Q_x = \int_{-h/2}^{h/2} \tau_{zx}\,dz,\ Q_y = \int_{-h/2}^{h/2} \tau_{yz}\,dz \qquad (6.8)$$

During vibration, suppose $w(x,y,t)$ be the deflection of the plate and $q(x,y,t)$ be the intensity of the load on the surface $z = -h/2$ at time t. Then vertical resolution of the forces gives the equation of motion as

$$\frac{\partial Q_x}{\partial x}dxdy + \frac{\partial Q_y}{\partial y}dxdy + q\ dxdy = \rho h dxdy\frac{\partial^2 w}{\partial t^2}$$

i.e. $$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \varepsilon = \rho h\frac{\partial^2 w}{\partial t^2}, \qquad (6.9)$$

ρ being the density of the plate. Now the D'Alembert's force (reversed effective force) $-\rho h dx dy \frac{\partial^2 w}{\partial t^2}$, the external forces and internal forces acting on the plate are in equilibrium. Taking moments of all these forces acting on the element w.r.t the x-axis, we have for equilibrium

$$\frac{\partial M_{xy}}{\partial x} dx dy + \frac{\partial M_y}{\partial y} dx dy - Q_y dx dy = 0$$

i.e. $$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \qquad (6.10a)$$

In which the moments of D'A lembert's force, load and the change in Q_y are neglected as being small quantities of higher order. Similarly, taking moments about the y-axis, we have

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}. \qquad (6.10b)$$

Substituting (6.10) into (6.9), we get

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \varepsilon = \rho h \frac{\partial^2 w}{\partial t^2}$$

which with the help of (6.6) reduces to

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = q \qquad (6.11)$$

where the flexural rigidity D is assumed to be constant.

Boundary Conditions :

(a) **Clamped edge :** $(w)_{x=a} = \left(\frac{\partial w}{\partial x}\right)_{x=a} = 0$

(b) **Simply supported edge :** $(w)_{x=a} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=a} = 0$

(c) **Free edge :** $\left[\frac{\partial^3 w}{\partial x^3} + (2-\sigma)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=a} = 0$

Initial Conditions :

These conditions give the initial deflection of the plate and its initial velocity so that

$$w = f(x,y),\ \frac{\partial w}{\partial t} = \phi(x,y) \text{ at } t = 0.$$

6.3 Vibration of a Rectangular Plate with Simply Supported Edge

Consider a rectangular plate of length a and breadth b with no streching of the middle surface plane. The equation of transverse vibration of the plate is given by (6.11) viz.

$$D\nabla^4 w(x,y,t) + \rho\frac{\partial^2 w(x,y,t)}{\partial t^2} = q(x,y,t). \tag{6.12}$$

For simply supported edge, the boundary conditions are

$$\text{(i) } w = 0,\ \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0, a \tag{6.13}$$

$$\text{and (ii) } w = 0,\ \frac{\partial^2 w}{\partial y^2} = 0 \text{ at y} = 0, b.$$

To solve the equation (6.12), we assume

$$w(x,y,t) = \bar{w}(x,y)e^{ipt},\ q(x,y,t) = \tilde{q}(x,y)e^{ipt} \tag{6.14}$$

which when substituted in (6.12) give

$$\nabla^4\bar{w} - \lambda^4\bar{w} = \frac{\tilde{q}}{D} \tag{6.15}$$

where

$$\lambda^4 = \frac{p^2}{c^2},\ c^2 = \frac{D}{\rho h}. \tag{6.16}$$

Let us represent $\bar{w}(x,y)$ and $\tilde{q}(x,y)$ in the form of double trigonometric series as

$$\bar{w}(x,y) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right).\sin\left(\frac{n\pi y}{b}\right) \tag{6.17a}$$

$$\text{and } \tilde{q}(x,y) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn} \sin\left(\frac{m\pi x}{a}\right).\sin\left(\frac{n\pi y}{b}\right) \tag{6.17b}$$

whose finite sine transforms, by inversion theorem, are

$$A_{mn} = \int_0^a dx \int_0^b \bar{w}(x,y)\sin\left(\frac{m\pi x}{a}\right).\sin\left(\frac{n\pi y}{b}\right)dy. \tag{6.18a}$$

$$Q_{mn} = \int_0^a dx \int_0^b \bar{q}(x,y)\sin\left(\frac{m\pi x}{a}\right).\sin\left(\frac{n\pi y}{b}\right)dy. \tag{6.18b}$$

It can be easily verified that all the boundary conditions in (6.13) are satisfied by the assumptions (6.17).

We now introduce the eigen function

$$W_{mn}(x,y) = \frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right).\sin\left(\frac{n\pi y}{b}\right) \tag{6.19}$$

Note that the boundary conditions in (6.13) are all satisfied by this function. Now if (6.19) represents m, n mode of free vibration of the plate with the corresponding eigenvalue $\lambda = \lambda_{mn}$ then it must satisfy the differential equation

$$\nabla^4 W_{mn} - \lambda_{mn}^4 W_{mn} = 0 \tag{6.20}$$

with the corresponding frequency p_{mn} given by

$$p_{mn}^2 = c^2\lambda_{mn}^4,\ c^2 = \frac{D}{\rho h}. \tag{6.21}$$

It is to be noted that the eigen functions introduced in (6.19) form a normalised othogonal set and satisfy the following orthogonal properties :

$$\int_0^a\int_0^b W_{ij}(x,y)W_{kl}(x,y)\,dxdy = 0,\ \text{if}\ \lambda_{ij} \neq \lambda_{kl}$$

and $$\int_0^a\int_0^b W_{ij}^2(x,y)\,dxdy = 1. \tag{6.22}$$

In terms of $W_{mn}(x,y)$, equations (6.17) lead to

$$\bar{w}(x,y) = \frac{2}{\sqrt{ab}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}W_{mn}(x,y), \tag{6.23}$$

$$\bar{q}(x,y) = \frac{2}{\sqrt{ab}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn}W_{mn}(x,y),$$

where

$$A_{mn} = \frac{\sqrt{ab}}{2}\int_0^a dx\int_0^b \bar{w}(x,y)W_{mn}(x,y)dy, \tag{6.24}$$

$$Q_{mn} = \frac{\sqrt{ab}}{2}\int_0^a dx\int_0^b \bar{q}(x,y)W_{mn}(x,y)dy,$$

To obtain A_{mn} in terms of Q_{mn}, we multiply both sides of the equation (6.15) by $W_{mn}(x,y)$ and integrate over the area of the plate to obtain

$$\int_0^a\int_0^b\left[\nabla^4\bar{w}(x,y)-\lambda^4\bar{w}(x,y)\right]W_{mn}(x,y)dxdy$$

$$=\frac{1}{D}\int_0^a\int_0^b\bar{q}(x,y)W_{mn}(x,y)dxdy. \tag{6.25}$$

Now $\int_0^a\int_0^b\nabla^4\bar{w}(x,y)W_{mn}(x,y)dxdy$

$$=\frac{2}{\sqrt{ab}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\int_0^a\int_0^b\left(\nabla^4W_{mn}\right).W_{mn}dx\,dy \quad \text{(by (6.23))}$$

$$=\frac{2}{\sqrt{ab}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\int_0^a\int_0^b\lambda^4_{mn}W^2_{mn}\,dx\,dy \quad \text{(by (6.20))}$$

$$=\frac{2}{\sqrt{ab}}\lambda^4_{mn}A_{mn} \quad \text{(by (6.22))}.$$

Also $\int_0^a\int_0^b\lambda^4\bar{w}(x,y)W_{mn}(x,y)dx\,dy$

$$=\frac{2}{\sqrt{ab}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda^4A_{mn}\int_0^a\int_0^b W^2_{mn}(x,y)\,dy$$

$$=\frac{2}{\sqrt{ab}}\lambda^4A_{mn}$$

and similarly, $\frac{1}{D}\int_0^a\int_0^b\bar{q}(x,y)W_{mn}(x,y)dx\,dy=\frac{2}{\sqrt{ab}}.\frac{Q_{mn}}{D}.$

Hence from (6.25) we get

$$\frac{2}{\sqrt{ab}}(\lambda_{mn}^4-\lambda^4)A_{mn}=\frac{2}{\sqrt{ab}}\frac{Q_{mn}}{D};\ \text{i.e.}\ A_{mn}=\frac{Q_{mn}}{D\left(\lambda_{mn}^4-\lambda^4\right)}.$$

Substituting this in (6.23) and using the relation (6.24) we have

$$\tilde{w}(x,y)=\frac{4}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_{mn}^4-\lambda^4}\sin\left(\frac{m\pi x}{a}\right).\sin\left(\frac{n\pi y}{b}\right)\times$$

$$\int_0^a\int_0^b\tilde{q}(\xi,\eta)\sin\left(\frac{m\pi\xi}{a}\right).\sin\left(\frac{n\pi\eta}{b}\right)d\xi\,d\eta. \tag{6.26}$$

Now from (6.19) and (6.20) we obtain

$$\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2=\lambda_{mn}^4=\frac{p_{mn}^2}{c^2}$$

Noting that $\left(\lambda_{mn}^4-\lambda^4\right)c^2=p_{mn}^2-p^2$, it follows from (6.26)

$$\tilde{w}(x,y)=\frac{4c^2}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{p_{mn}^2-p^2}\sin\left(\frac{m\pi x}{a}\right).\sin\left(\frac{n\pi y}{b}\right)\times$$

$$\times\int_0^a\int_0^b\tilde{q}(\xi,\eta)\sin\left(\frac{m\pi\xi}{a}\right).\sin\left(\frac{n\pi\eta}{b}\right)d\xi\,d\eta \tag{6.27}$$

and finally, the deflection $w(x,y,t)$ is given by

$$w(x,y,t)=\tilde{w}(x,y)e^{ipt}. \tag{6.28}$$

6.4 Free Vibration of a Circular Plate

In two-dimensional polar coordinates (r,θ), the differential equation of transverse vibration of a plate given by (6.11) reduces to

$$c^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 w+\frac{\partial^2 w}{\partial t^2}=\frac{q(r,\theta,t)}{\rho h} \tag{6.29}$$

where $w=w(r,\theta,t)$ is the deflection of the plate, $c^2=D/\rho h$, ρ is the uniform density, h is the plate thickness and D is the constant flexural rigidity of the plate.

The bending and twisting moments are given from (6.6) as

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right],$$

$$M_\theta = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \sigma\frac{\partial^2 w}{\partial r^2}\right], \tag{6.30}$$

$$M_{r\theta} = -D(1-\sigma)\left[\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right].$$

Assuming

$$w(r,\theta,t) = \tilde{w}(r,\theta)e^{ipt}, \tag{6.31}$$

and noting that for free vibration $q(r,\theta,t) = 0$, we get from (6.29)

$$\nabla^4 \tilde{w} - \lambda^4 \tilde{w} = 0 \tag{6.32}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ and $\lambda^2 = p/c$. The general solution of the equation (6.32) finite at all points of the plate is

$$\tilde{w}(r,\theta) = {\cos \atop \sin}(m\theta)\left[AJ_m(\lambda r) + BI_m(\lambda r)\right] \tag{6.33}$$

where J_m is Bessel function and I_m is modified Bessel function of first kind of order m and A, B are constants.

We now consider the following cases :

Case - I : Clamped edge

In this case $\tilde{w} = 0, \frac{\partial \tilde{w}}{\partial r} = 0$ at $r = a$.

The condition $\tilde{w} = 0$ at $r = a$ gives from (6.33)

$$B = -A.\frac{J_m(\lambda a)}{I_m(\lambda a)}. \tag{6.34}$$

Also, use of the recurrence relations for Bessel functions

$$J'_m(\lambda r) = \tfrac{1}{2}\left[J_{m-1}(\lambda r) - J_{m+1}(\lambda r)\right],$$

$$I'_m(\lambda r) = \tfrac{1}{2}\left[I_{m-1}(\lambda r) + I_{m+1}(\lambda r)\right], \tag{6.35}$$

the second boundary condtion $\frac{\partial \bar{w}}{\partial r} = 0$ at $r = a$ gives from (6.33)

$$A\left[J_{m-1}(\lambda a) - J_{m+1}(\lambda a)\right] + B\left[I_{m-1}(\lambda a) + I_{m+1}(\lambda a)\right] = 0$$

which with the help of (6.34) gives the frequency equation as

$$I_m(\lambda a)\left[J_{m-1}(\lambda a) - J_{m+1}(\lambda a)\right] = J_m(\lambda a)\left[I_{m-1}(\lambda a) + I_{m+1}(\lambda a)\right] \qquad (6.36)$$

This equation can be expressed as a power series in λa involving m. If only the first term is retained, we get a value of λa which we denote by $\lambda_{1m} a$; if first two terms are retained, we get $\lambda_{2m} a$ and so on. In this way consecutive values of λ_{nm} are obtained. Noting that

$$p_{nm} = \frac{1}{a^2}(\lambda_{nm} a)^2 \sqrt{D/\rho h}, \quad \left[\because p = \lambda^2 c = \lambda^2 \sqrt{D/\rho h}\right] \qquad (6.37)$$

we see that p_{nm} is known and so are the frequencies of the various modes of free vibrations given by

$$\bar{w}_{nm}(r,\theta) = A_{nm} \begin{matrix}\cos\\ \sin\end{matrix}(m\theta)\left[J_m(\lambda_{nm} r) - \frac{J_m(\lambda_{nm} a)}{I_m(\lambda_{nm} a)} I_m(\lambda_{nm} r)\right] \qquad (6.38)$$

Case - II : Simply supported edge

In this case, the boundary conditions at $r=a$ are

$$w = 0 \text{ and } M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] = 0$$

Assuming

$$w(r,\theta,t) = \bar{w}(r,\theta)e^{ipt},$$

the above boundary conditions give

$$\bar{w} = 0, \frac{\partial^2 \bar{w}}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial \bar{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{w}}{\partial \theta^2}\right) = 0 \qquad (6.39)$$

Proceeding exactly along the same lines as in Case - I, we find that the frequency equation is

$$I_m(\lambda a)\left[(\lambda a)^2\{J_{m-2}(\lambda a) - 2J_m(\lambda a) + J_{m+2}(\lambda a)\}\right.$$
$$\left. + 2\sigma(\lambda a)\{J_{m-1}(\lambda a) - J_{m+1}(\lambda a)\} - 4\sigma m^2 J_m(\lambda a)\right]$$

$$J_m(\lambda a)\Big[(\lambda a)^2\{I_{m-2}(\lambda a)+2I_m(\lambda a)+I_{m+2}(\lambda a)\}$$

$$+2\sigma(\lambda a)\{I_{m-1}(\lambda a)-I_{m+1}(\lambda a)\}-4\sigma m^2 I_m(\lambda a)\Big] \tag{6.40}$$

From this, we can evaluate consecutive values of λ_{mn} and hence frequencies of various modes of vibration given by

$$\bar{w}_{mn}(r,\theta)=A_{mn}\begin{matrix}\cos\\ \sin\end{matrix}(m\theta)\left[J_m(\lambda_{mn}r)-\frac{J_m(\lambda_{mn}a)}{I_m(\lambda_{mn}a)}I_m(\lambda_{mn}r)\right]. \tag{6.41}$$

6.5 Symmetrical Vibrations of a Thin Circular Plate

For axisymmetric modes of vibration of a thin circular plate of radius a, the deflection w is independent of θ. We suppose that the external force on the plate is symmetrically placed. Then the equation of vibration (6.11) takes the form

$$c^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2 w+\frac{\partial^2 w}{\partial t^2}=\frac{q(r,t)}{\rho h} \tag{6.42}$$

where $c^2 = D/\rho h$ and $w = w(r,t)$. If the plate be simply supported on the boundary $r=a$, then the boundary conditions are

$$w=0,\ M_r=-D\left(\frac{\partial^2 w}{\partial r^2}+\frac{\sigma}{r}\frac{\partial w}{\partial r}\right)=0 \text{ at } r=a \tag{6.43}$$

Sneddon has slightly modified the second condition as

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}=0 \text{ at } r=a \tag{6.43'}$$

and noted that this modification has no appreciable effect on the final solutions.

To solve the equation (6.42), we put

$$w(r,t)=\bar{w}(r)e^{ipt},\ q(r,t)=\bar{q}(p)e^{ipt}. \tag{6.44}$$

Then the equation (6.42) reduces to

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)^2\bar{w}-\lambda^4\bar{w}=\frac{\bar{q}}{D} \tag{6.45}$$

where $\lambda^4 = p^2/c^2$.

Now

$$\int_0^a r\left(\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\frac{d\bar{w}}{dr}\right)J_0(\alpha r)dr=\int_0^a \frac{d}{dr}\left(r\frac{d\bar{w}}{dr}\right).J_0(\alpha r)dr$$

$$=\left[r\frac{d\bar{w}}{dr}J_0(\alpha r)-\alpha r\bar{w}J_0'(\alpha r)\right]_0^a-\alpha^2\int_0^a r\,\bar{w}(r).J_0(\alpha r)dr \qquad (6.46)$$

The expression within the bracket on the R.H.S. vanishes if $J_0(\alpha a)=0$, if α_i (i=1, 2,3,...) is a root of the transcendental equation $J_0(\alpha_i a)=0$. Hence replacing α by α_i in (6.46), we get

$$\int_0^a r\left(\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\frac{d\bar{w}}{dr}\right)J_0(\alpha_i r)dr=-\alpha_i^2\int_0^a r\,\bar{w}(r).J_0(\alpha_i r)dr$$

Using this and the boundary condition (6.43), we have

$$\int_0^a r\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\frac{d\bar{w}}{dr}\right)J_0(\alpha_i r)dr=\alpha_i^4\int_0^a r\,\bar{w}(r).J_0(\alpha_i r)dr \qquad (6.47)$$

Let us now define finite Hankel transforms of $\bar{w}(r)$ and $\bar{q}(r)$ by

$$\bar{\bar{w}}(\alpha_i)=\int_0^a r\bar{w}(r).J_0(\alpha_i r)dr,\ \bar{\bar{q}}(\alpha_i)=\int_0^a r\bar{q}(r).J_0(\alpha_i r)dr. \qquad (6.48a,b)$$

Multiplying both sides of (6.45) by $J_0(\alpha_i r)$ and integrating both sides w.r.t. r between 0 to a, we obtain

$$\bar{\bar{w}}(\alpha_i)=\frac{\bar{\bar{q}}(\alpha_i)}{D\left(\alpha_i^4-\lambda^4\right)} \qquad (6.49)$$

Now the inverse Hankel transform of (6.48a) gives

$$\bar{w}(r)=\frac{2}{a^2}\sum_{i=1}^{\infty}\bar{\bar{w}}(\alpha_i)\frac{J_0(\alpha_i r)}{\left[J_1(\alpha_i a)\right]^2}$$

Hence, using (6.49) we obtain

$$\bar{w}(r)=\frac{2}{a^2D}\sum_{i=1}^{\infty}\frac{J_0(\alpha_i r)}{\left(\alpha_i^4-\lambda^4\right)\left[J_1(\alpha_i a)\right]^2}\int_0^a \xi\bar{q}(\xi)J_0(\alpha_i\xi)d\xi \qquad (6.50)$$

and finally

$$w(r,t)=\bar{w}(r)e^{i\sigma t}. \qquad (6.51)$$

6.6 Summary

Assuming small deflection theory, transverse vibration of thin elastic plates has been considered in this Unit. Differential equation of vibration is obtained and the different types of edge condition are noted. As applications, transverse vibrations of thin rectangular and circular plates are considered.

6.7 Exercises

1. ***Short Answer Type :***

   (a) State the assumptions made in the small deflection theory of thin elastic plates.

   (b) Write down the expressions for displacements, strains and stresses in terms of deflection.

   (c) Find the bending and twisting moments of a thin elastic plate.

   (d) Define the flexural (bending) rigidity.

   (e) Write down different types of edge conditions of a thin circular plate.

2. ***Broad Answer Type :***

   (a) Deduce the equation of transverse vibration of a thin elastic plate in small deflection theory.

   (b) Solve the problem of vibration of a thin rectangular plate with simply supported edge.

   (c) Solve the problem of free vibration of a thin circular plate with (i) clamped edge, (ii) simply supported edge.

   (d) Using Hankel transform method, obtain the deflection for symmetrical vibration of a thin circular plate.

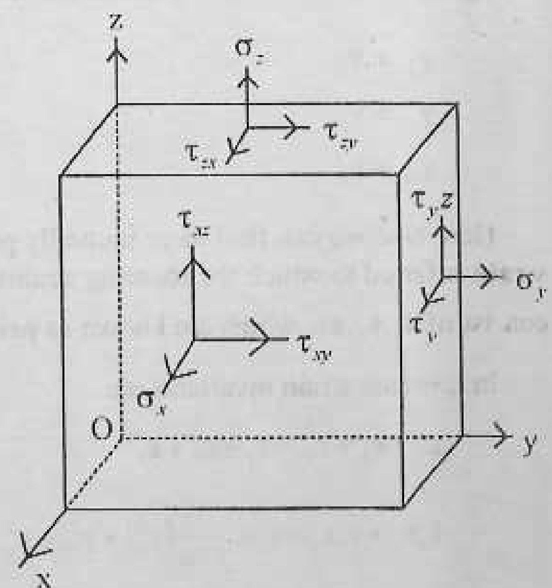
Unit-7 ☐ Plasticity

(In this Unit we shall use unabridged notations in place of tensor notations)

Structure

7.1 Basic Concepts

The components of stress at a point 0 are given by $\sigma_x, \tau_{xy}, \tau_{xz}$ across a plane whose normal is parallel to x-axis while $\tau_{yx}, \sigma_y, \tau_{yz}$ across the plane with normal parallel to y-axis and $\tau_{zx}, \tau_{zy}, \sigma_z$ across the plane having normal parallel to z-axis. $\sigma_x, \sigma_y, \sigma_z$ are normal components of stress and the other components are the shearing stresses. Six components of shearing stress satisfy the relation



$$\tau_{pq} = \tau_{qp}, \ (p, q = x, y, z \text{ and } p \neq q) \quad (7.1)$$

There are three mutually perpendicular directions called **principal axes of stress** in which the stress is purely normal and has the values $\sigma_1, \sigma_2, \sigma_3$ called the **principal stresses** and for which shearing stresses are zero. Also there are three quantities

I_1, I_2, I_3 which remain unchanged or **invariant** by transformation from one orthogonal set of axes to another given by

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 \tag{7.2}$$

$$\begin{aligned} I_2 &= -\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) + \tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2 \\ &= -\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \end{aligned} \tag{7.3}$$

$$\text{or,} \quad -I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_z & \tau_{zx} \\ \tau_{zx} & \sigma_x \end{vmatrix}$$

and

$$\begin{aligned} I_3 &= \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} \\ &= \sigma_x\sigma_y\sigma_z + 2\tau_{yz}\tau_{zx}\tau_{xy} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx} - \sigma_z\tau_{xy}^2 \end{aligned} \tag{7.4}$$

Deformation is specified by six components of strain namely $\varepsilon_x, \varepsilon_y, \varepsilon_z$ which are analogous in the direction of the axes *x, y, z* and three shearing strain

$$\begin{aligned} \gamma_{xy} &= \gamma_{yx} \\ \gamma_{yz} &= \gamma_{zy} \\ \gamma_{zx} &= \gamma_{xz} \end{aligned} \tag{7.5}$$

Here also we can find three mutually perpendicular lines called **principal axes of strain** referred to which the shearing strains vanish. The strain components in this case consist of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, which are known as **principal strains.**

In this case strain invariants are

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{7.6}$$

$$\begin{aligned} \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}\left(\gamma_{yz}^2 + \gamma_{zx}^2 + \gamma_{xy}^2\right) \\ = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 \end{aligned} \tag{7.7}$$

$$\varepsilon_x\varepsilon_y\varepsilon_z + \frac{1}{4}\left(\gamma_{yz}\gamma_{zx}\gamma_{xy} - \varepsilon_x\gamma_{yz}^2 - \varepsilon_y\gamma_{zx}^2 - \varepsilon_z\gamma_{xy}^2\right)$$
$$= \varepsilon_1\varepsilon_2\varepsilon_3 \qquad (7.8)$$

Dilatation : Let $a, b, c,$ be the length of three elements along the principal aexs of strain. After deformation these lengths become $a(1+\varepsilon_1), b(1+\varepsilon_2), c(1+\varepsilon_3)$.

Then,

$$\text{Dilatation} = \Delta = \frac{\text{change in volume}}{\text{original volume}}$$

$$= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_x + \varepsilon_y + \varepsilon_z \qquad (7.9)$$

We shall use the symbol G for modules of rigidity and ν for the Poisson's ratio.

Stress deviator : The mean stress s is definied as

$$s = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad (7.10)$$

This is an invariant. The stress deviators s_x, s_y, s_z, s_{xy}, s_{yz}, s_{zx} are given by

$$s_x = \sigma_x - s,\ s_y = \sigma_y - s,\ s_z = \sigma_z - s$$
$$s_{xy} = \tau_{xy},\quad s_{yz} = \tau_{yz},\quad s_{zx} = \tau_{zx} \qquad (7.11)$$

If s_1, s_2, s_3 be components of the stress deviation along the principal axes then

$$s_1 = \sigma_1 - s = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3} \qquad (7.12)$$

Similarly

$$s_2 = \sigma_2 - s = \frac{2\sigma_2 - \sigma_3 - \sigma_1}{3} \qquad (7.13)$$

$$s_3 = \sigma_3 - s = \frac{2\sigma_3 - \sigma_1 - \sigma_2}{3} \qquad (7.14)$$

By adding we get,

$$s_1 + s_2 + s_3 = 0$$

Also $s_x + s_y + s_z = \sigma_x + \sigma_y + \sigma_z - \frac{\sigma_x + \sigma_y + \sigma_z}{3}.3 \qquad (7.15)$

$$= 0$$

Strain deviator :

In a similar way we have from the strain components the mean strain defined by

$$e = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3} = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} = \frac{\Delta}{3} \text{ which is an invariant.} \qquad (7.16)$$

The components of strain deviator

$e_x, e_y, e_z, e_{xy}, e_{yz}, e_{zx}$ can be written as

$$e_x = \varepsilon_x - e,\ e_y = \varepsilon_y - e,\ e_z = \varepsilon_z - e \qquad (7.17)$$

$$e_{yz} = \gamma_{yz},\ e_{zx} = \gamma_{zx},\ e_{xy} = \gamma_{xy}$$

so that

$$e_x + e_y + e_z = \varepsilon_x + \varepsilon_y + \varepsilon_z - 3e = 0$$

If e_1, e_2, e_3 be the components of principal strain deviator then

$$e_1 = \varepsilon_1 - e,\ e_2 = \varepsilon_2 - e,\ e_3 = \varepsilon_3 - e, \qquad (7.18)$$

so that

$$e_1 + e_2 + e_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 3e = 0 \qquad (7.19)$$

Relation between the stress and strain deviators :

For an isotropic material the stress-strain relations are :

$$\begin{aligned} \sigma_1 &= \lambda\Delta + 2G\varepsilon_1 \\ \sigma_2 &= \lambda\Delta + 2G\varepsilon_2 \\ \sigma_3 &= \lambda\Delta + 2G\varepsilon_3 \end{aligned} \qquad (7.20)$$

so that by adding we get

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\lambda\Delta + 2G\Delta = (3\lambda + 2G)\Delta$$

Let $\sigma_1 = \sigma_2 = \sigma_3 = -p$

$$\therefore \sigma_1 + \sigma_2 + \sigma_3 = -3p = 3(3\lambda + 2G)e$$

$$\therefore -\frac{p}{e} = (3\lambda + 2G) \qquad (7.21)$$

$$-\frac{p}{3e} = -\frac{p}{\Delta} = K = \text{Bulk modulus}$$

$$\therefore -\frac{p}{e} = (3\lambda + 2G) = 3K$$

Moreover, $\sigma_1+\sigma_2+\sigma_3=3s=-3p$

$$\therefore s=-p$$

so that $\frac{s}{e}=-\frac{p}{e}=3K$ (7.22)

and hence $s=3Ke$

Again we have

$$\sigma_x=\lambda\Delta+2G\varepsilon_x$$

$$\therefore s_x+s=3\lambda e+2G(e+e_x)=e(3\lambda+2G)+2Ge_x$$
$$=3Ke+2Ge_x$$
$$=s+2Ge_x$$

$$\left.\begin{array}{l}\therefore s_x=2Ge_x\\ \text{Similarly}\\ \quad s_y=2Ge_y\\ \quad s_z=2Ge_z\end{array}\right\} \quad (7.23)$$

Also

$$\begin{array}{l}s_{yz}=\tau_{yz}=G\gamma_{yz}=Ge_{yz}\\ s_{zx}=\tau_{zx}=G\gamma_{zx}=Ge_{zx}\\ s_{xy}=\tau_{xy}=G\gamma_{xy}=Ge_{xy}\end{array} \quad (7.24)$$

Thus components of stress deviator can be considered as the components of stress involving no dilatation.

Stress-strain curve :

Suppose that a rod of ductile metal is stressed in tension in a testing mechine. The observable quantities are the stress σ which is the load applied by the mechine divided by the original area of cross-section of the rod and the strain ε is given by

$$\varepsilon=\frac{l-l_0}{l_0}$$

where l_0 is the original length of the rod and l is the length when the stress is σ. If σ is plotted against ε a stress-strain curve (Fig.1) is obtained. From the law of stresses it is found that if the stress is reduced to zero the rod returns to its original length i.e. there is no permanent deformation. **This is the property of elasticity** and the range of stress in which there is no permanent deformation is called **elastic region**. In this region it is found that the stress is proportional to the strain. The stress σ_0 at which permanent deformation first appears is called the **yield stress** and the point is called **yield point**. It is at the point A corresponding to the stress σ_0 that the curvature of the stress curve is first noticed. Deformation at a stress above the yield stress is described as plastic deformation and a material with a yield stress may be idealised by the **perfectly plastic solid** (Fig. 2). The behaviour is elastic below the yield stress σ_0 but on attaining this stress, it flows plastically under the constant stress.

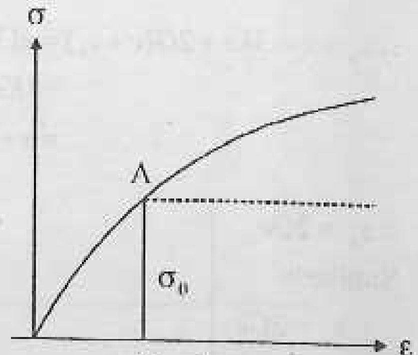


Fig. 1

During this flow we no longer have a one-to-one correspondance between stress and strain. If the stress has been maintained at the value σ_0 for any finite time before unloading, plastic flow occurs during this time and specimen will exibit a permanent deformation after unloading. The stress-strain relation during such an unloading is represented by the portion BC (Fig.2) and the permanent elongation by the segment OC. Then segment BC obtained during unloading is parallel to the segment OA obtained during the first loading. If the specimen is reloaded after being completely or partially unloaded the stress-strain diagram for reloading coincides with the diagram for the preceeding unloading until the critical stress is reached again. Thereafter the specimen flows plastically as if the unloading and reloading had never taken place. The stress-strain relation during such a reloading and subsequent plastic flow is represented by the portion CBD (Fig.2).

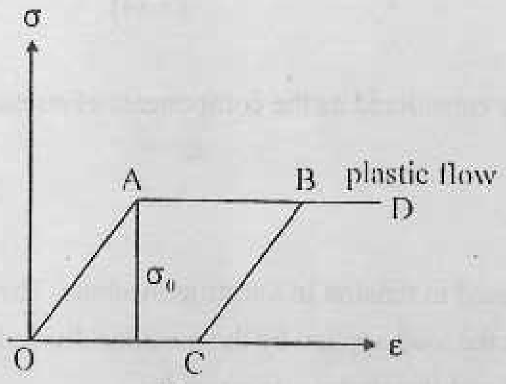


Fig. 2

The typical stress-strain diagram (Fig.3) shows a rise in region AD beyond the yield stress. If the material is stressed beyond the yield stress. If the material is stressed beyond the yield stress σ_0 at A and the load is then removed at a point D well beyond A, the strain is reduced to ε_s and not to zero value. ε_s is the permanent deformation. If now the load is reapplied a narrow loop EFG, with FG very nearly parallel to OA, will be described.

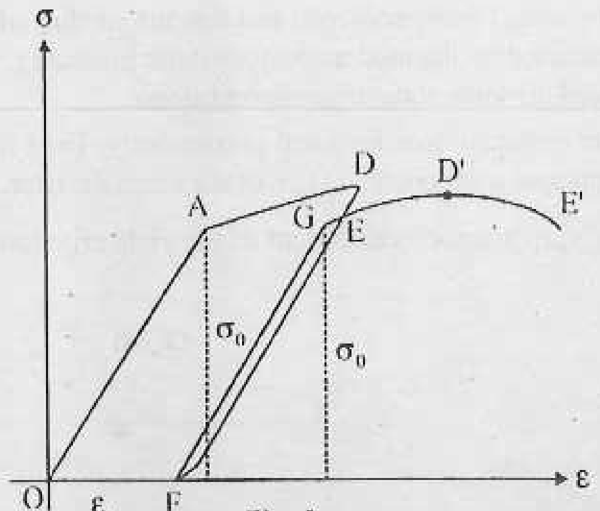


Fig. 3

The diagram indicates that following the cycle of unloading and the reapplication of the load the elastic strain range lead upto G which is higher then A. Thus the stress at which plasticity occur in this case $> \sigma_0$. This is the phenomenon of **work hardening** in which the material is in a sense stronger after the load cycle has taken place. Thus, we observe again an intervel of elastic strain with a new proportion limit σ_0 followed by an interval of small plastic strain. As the stress is brought near to the original value the curve bents sharply near D' and part D'E' becomes virtually a continuation of AD. Indeed, if the stress had been increased continuously from D the same curve DE' would have been described.

7.2 Yield Criterion

The stress-strain curve described above is only for the uniaxial state of stress. It is important to know the behaviour of a material under combined stresses. In particular, it is necessary to have an idea of the conditions which characterised the transition of material from elastic state to the state of yielding. In simple tension σ = constant $= \sigma_0$ in the state of yielding and in simple shear $\tau = \text{constant} = \tau_0$.

The question arises here as to the possible form of the conditions characterising the transition beyond the elastic limit under combined stresses. This condition which is to be fulfilled in the state of yielding is called **yield condition**.

For an isotropic body, since the plastic yielding depends only on the magnitude of the three principal applied stresses and not on their direction any yielding criteria can be expressed in the form

$$f(I_1, I_2, I_3) = 0 \qquad (7.25)$$

where I_1, I_2 and I_3 are first three invariants of the stress tensor. An immediate simplification of (7.25) can be achived by using the experimental fact that the yielding of a metal is, to a first approximation, unaffected by the moderate hydrostatic pressure or tension either applied alone or superposed on some state of combined stress.

Suppose that the above observation is strictly true for ideal plastic body. Then it follows that yielding depends only on principal components s_1, s_2, s_3 of the stress deviator.

Since $s_1 + s_2 + s_3 = 0$ implies that s_1, s_2, s_3 are not independent so the yield criterion can be reduced to the form

$$f(I_2', I_3') = 0 \qquad (7.26)$$

where $I_2' = -(s_1 s_2 + s_2 s_3 + s_3 s_1)$

$I_3' = s_1 s_2 s_3$

There are two important theorems available to predict the begining of the plastic yielding in ductile metals.

(a) Maximum shearing stress or Tresca's criterion

The criterion states that yielding occurs at a point when the magnitude of the maximum shearing stress has the value $\frac{1}{2}\sigma_0$ which is a constant for a material.

If $\sigma_1, \sigma_2, \sigma_3$ be the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$ then it is known that the gratest shearing stress is $\frac{1}{2}(\sigma_1 - \sigma_3)$ and it acts across the plane whose normal bisects the angle between the directions of the greatest and least principal axes. Thus, if $\sigma_1 > \sigma_2 > \sigma_3$, Tresca's condition can put as

$$\tfrac{1}{2}(\sigma_1 - \sigma_3) = \tfrac{1}{2}\sigma_0$$

i.e. $\sigma_1 - \sigma_3 = \sigma_0$ (7.27)

In order to use (7.27) it is necessary to know the principal stresses and also the knowledge of greatest and least principal stresses. For simple cases it may be possible to find, but in many cases it is difficult to obtain the information. Hence (7.27) is not quite suitable as general mathematical formulation.

(b) The criterion of Von Mises :

Frem (7.26) we see that the yield criterion can be written in the form

$$f\left(I_2', I_3'\right) = 0$$

According to Von Mises, yielding occurs when I_2' reaches the critical value, a constant of the material. The criterion can be written in the alternative form as

$$s_1^2 + s_2^2 + s_3^2 = \frac{2\sigma_0^2}{3} \tag{7.28}$$

or, $2s_1^2 + 2s_2^2 + 2s_3^2 - 2s_1s_2 - 2s_2s_3 - 2s_3s_1 + s_1^2 + s_2^2 + s_3^2$

$$+2s_1s_2 + 2s_2s_3 + 2s_3s_1 = 2\sigma_0^2$$

or, $(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2 + (s_1 + s_2 + s_3)^2 = 2\sigma_0^2$

or, $(s_1 - s_2)^2 + (s_2 - s)^2 + (s_3 - s_1)^2 = 2\sigma_0^2$

and noting that $s_i = \sigma_i - s, \; i = 1,2,3.$

We have

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2 \tag{7.29}$$

(7.28) and (7.29) are the two alternative forms of Von Mises criteria.

Corollary 1 : When $\sigma_2 = \sigma_3 = 0$, Von Mises Criteria becomes $\sigma_1 = \sigma_0$.

Thus σ_0 is the yield stress in uniaxial tension.

Corollary 2 : Interpretation of Mises Criteria based on strain energy.

If there be principal extension $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and corresponding principal stresses are $\sigma_1, \sigma_2, \sigma_3$, then the strain energy W per unit volume is given by

$$2W = \sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3.$$

Since $s_i = \sigma_i - s$ and $e_i = \varepsilon_i - e$ for $i = 1,2,3$.

We have

$$2W = \sum_{i=1}^{3} (s_i + s)(e_i + e)$$

$$= s_1e_1 + s_2e_2 + s_3e_3 + 3se$$

$$+e(s_1 + s_2 + s_3) + s(e_1 + e_2 + e_3)$$

Noting that $\sum s_i = 0 = \sum e_i$, $s_i = 2Ge_i$, $\quad s = 3Ke$,

it follows that $2W = \frac{1}{2G}\left(s_1^2 + s_2^2 + s_3^2\right) + \frac{s^2}{K}$.

Now the components of stress deviator are given by

$$s_i = 2Ge_i$$

Since $\sum e_i = 0$ it follows that, the component of stress deviator can be considered as component which produces no dilatation. Hence whatever dilatation be produced is due to mean stresses only. Thus, if the mean stress acting in all directions produces the principal strains $\varepsilon_1, \varepsilon_2, \varepsilon_3$, the corresponding energy of deformation W_s is given by

$$2W_s = s\varepsilon_1 + s\varepsilon_2 + s\varepsilon_3 = 3se$$

$$= \frac{s^2}{K} \left[\text{since } s = 3Ke\right].$$

Therefore,

$$W = \frac{1}{4G}\left[s_1^2 + s_2^2 + s_3^2\right] + W_s.$$

Thus it follows that the elastic strain energy per unit volume can be splite into a part W_s associated with change in volume and a part $W = \frac{1}{4G}\left[s_1^2 + s_2^2 + s_3^2\right]$ associated with distortion.

Thus Von Mises Criterian $s_1^2 + s_2^2 + s_3^2 =$ constant for yielding is equivalent to the statement that **the strain energy of distortion attains the constant value characteristic of the material.**

7.3 Equations of Plasticity

I. Prandtl - Reuss Theory :

In the plastic range the total strain may be considered as the sum of the elastic strain and the plastic or permanent strain. The mean normal strain and the strain deiator may be decomposed in the same manner into the elastic and plastic components. Here single prime will be used to denote elasitc components and double prime to denote the plastic components. In atmost all the theories of plasticity it is assumed that there is no permanent change in volume. This means that the plastic mean normal strain e must vanish. i.e.

$$e'' = \frac{1}{3}\left(\varepsilon_x'' + \varepsilon_y'' + \varepsilon_z''\right) = \frac{1}{3}\left(\varepsilon_1'' + \varepsilon_2'' + \varepsilon_3''\right)$$

$$\therefore e'' = 0 \qquad (7.30)$$

we know that,

$$e_x'' = \varepsilon_x'' - e''.$$

so that $e_x'' = \varepsilon_x''$

similarly $e_y'' = \varepsilon_y'',\ e_z'' = \varepsilon_z''$ (7.31)

In other words, the plastic strain deviator is identical with plastic strain.

Furthermore, all these theories assume that during plastic flow the rate of change of plastic strain (or plastic strain deviator) is at any instant proportional to the instantaneous stress deviator. To be able to combine the equations expressing these assumptions with those of the rate of change of elastic strain we write them in the form

$$\begin{array}{lll|lll} 2G\dot{e}_x'' & = & \lambda s_x & G\dot{\gamma}_{yz}'' & = & \lambda\tau_{yz} \\ 2G\dot{e}_y'' & = & \lambda s_y & G\dot{\gamma}_{zx}'' & = & \lambda\tau_{zx} \\ 2G\dot{e}_z'' & = & \lambda s_z & G\dot{\gamma}_{xy}'' & = & \lambda\tau_{xy} \end{array} \qquad (7.32)$$

where λ is a positive factor of proportionality. According to Hooke's law the rate of change of elastic strain deviator is

$$\begin{array}{lll|lll} 2G\dot{e}_x' & = & \dot{s}_x & G\dot{\gamma}_{yz}' & = & \dot{\tau}_{yz} \\ 2G\dot{e}_y' & = & \dot{s}_y & G\dot{\gamma}_{zx}' & = & \dot{\tau}_{zx} \\ 2G\dot{e}_z' & = & \dot{s}_z & G\dot{\gamma}_{xy}' & = & \dot{\tau}_{xy} \end{array} \qquad (7.33)$$

Combining (7.32) and (7.33) we obtain, the following relation for the rate of total strain

$$\begin{array}{lll|lll} 2G\dot{e}_x & = & \dot{s}_x + \lambda s_x & G\dot{\gamma}_{yz} & = & \dot{\tau}_{yz} + \lambda\tau_{yz} \\ 2G\dot{e}_y & = & \dot{s}_y + \lambda s_y & G\dot{\gamma}_{zx} & = & \dot{\tau}_{zx} + \lambda\tau_{zx} \\ 2G\dot{e}_z & = & \dot{s}_z + \lambda s_z & G\dot{\gamma}_{xy} & = & \dot{\tau}_{xy} + \lambda\tau_{xy} \end{array} \qquad (7.34)$$

where $\dot{e}_x = \dot{e}_x' + \dot{e}_x''$; $\dot{\gamma}_{yz} = \dot{\gamma}_{yz}' + \dot{\gamma}_{yz}''$ etc.

These equations, however, are applied during plastic flow when the yield conditions requires that.

$$J_2 = -(s_x s_y + s_y s_z + s_z s_x) + \tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2$$
$$= \tfrac{1}{2}(s_x^2 + s_y^2 + s_z^2) + \tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2$$
$$= \text{constant} = \cdot k^2. \qquad (7.35)$$

Then,

$$\dot{J}_2 = s_x \dot{s}_x + s_y \dot{s}_y + s_z \dot{s}_z + 2\tau_{yz}\dot{\tau}_{yz} + 2\tau_{zx}\dot{\tau}_{zx} + 2\tau_{xy}\dot{\tau}_{xy} = 0 \qquad (7.36)$$

Let us assume

$$\dot{W} = s_x \dot{e}_x + s_y \dot{e}_y + s_z \dot{e}_z + \tau_{yz}\dot{\gamma}_{yz} + \tau_{zx}\dot{\gamma}_{zx} + \tau_{xy}\dot{\gamma}_{xz} \qquad (7.37)$$

which can be computed whenever the stress and the rate of strain are known. From equation (7.35) and (7.37) we have

$$2\dot{W}G = \dot{J}_2 + 2J_2$$
$$= 0 + 2k^2\lambda$$
$$\therefore \lambda = \frac{G\dot{W}}{k^2} \qquad (7.38)$$

Since λ has been defined as a positive quantity, $\dot{W}$ must be positive during the plastic flow. With the above value of λ we have

$$\begin{array}{l|l} \dot{s}_x = 2G\left(\dot{e}_x - \dfrac{\dot{W}}{2k^2} s_x\right) & \dot{\tau}_{yz} = G\left(\dot{\gamma}_{yz} - \dfrac{\dot{W}}{k^2}\tau_{yz}\right) \\ \dot{s}_y = 2G\left(\dot{e}_y - \dfrac{\dot{W}}{2k^2} s_y\right) & \dot{\tau}_{zx} = G\left(\dot{\gamma}_{zx} - \dfrac{\dot{W}}{k^2}\tau_{zx}\right) \\ \dot{s}_z = 2G\left(\dot{e}_z - \dfrac{\dot{W}}{2k^2} s_z\right) & \dot{\tau}_{xy} = G\left(\dot{\gamma}_{xy} - \dfrac{\dot{W}}{k^2}\tau_{xy}\right) \end{array} \qquad (7.39)$$

when the state of stress which fulfills the yield conditions $J_2 = k^2$, is given and a strain rate together with given state of stress furnishes the value of $\dot{W}$, the equation (7.39) determines the rate of change of stress deviator.

Corollary : Interpretation of $\dot{W}$:

Let W_1 be the strain energy of deformation per unit volume. Now, W_1 is a function of strain components $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ and so

$$\frac{dW_1}{dt} = \frac{\partial W_1}{\partial \varepsilon_x}\frac{\partial \varepsilon_x}{\partial t} + \frac{\partial W_1}{\partial \varepsilon_y}\frac{\partial \varepsilon_y}{\partial t} + \frac{\partial W_1}{\partial \varepsilon_z}\frac{\partial \varepsilon_z}{\partial t} + \frac{\partial W_1}{\partial \gamma_{xy}}\frac{\partial \gamma_{xy}}{\partial t} + \ldots$$

$$= \sigma_x \dot{\varepsilon}_x + \sigma_y \dot{\varepsilon}_y + \sigma_z \dot{\varepsilon}_z + \tau_{xy}\dot{\gamma}_{xy} + \ldots\ldots$$

$$= \sigma_x \left(\dot{e}_x + \dot{e}\right) + \ldots\ldots\ldots + \tau_{xy}\dot{\gamma}_{xy} + \ldots\ldots \quad (7.40)$$

(Since $e_x = \varepsilon_x - e$, etc.)

$$= e(\sigma_x + \sigma_y + \sigma_z) + \sigma_x \dot{e}_x + \sigma_y \dot{e}_y + \ldots\ldots + \tau_{xy}\dot{\gamma}_{xy} + \ldots \quad (7.41)$$

The first term in (7.41)

$$= 3se = s\frac{d}{dt}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

= The rate of work done in producing dilatation by means of normal stress operating in all directions.

The remaining term in (7.41)

$$= (s_x + s)\dot{e}_x + \ldots\ldots + \tau_{xy}\dot{\gamma}_{xy} + \ldots\ldots \text{since } s_x = \sigma_x - s$$

$$= s(\dot{e}_x + \dot{e}_y + \dot{e}_z) + s_x \dot{e}_x + \ldots\ldots + \tau_{xy}\dot{\gamma}_{xy} + \ldots\ldots \text{since } e_x = \varepsilon_x - e \text{ etc.} \quad (7.42)$$

$$\therefore e_x + e_y + e_z = \varepsilon_x + \varepsilon_y + \varepsilon_z - 3e = 0.$$

Hence the first expression in (7.42) is zero. Therfore from (7.41) and (7.42) we have

$$\frac{dW_1}{dt} = \text{Terms giving rate of change of volume}$$

$+ \dot{W}$ (Containing terms producing distorsion only).

Thus, we conclude that

$\dot{W}$ = The rate at which strain energy of deformation changes due to distortion only.

II. Stress-strain relation of Von-Mises :

In many problem of unrestricted plastic flow the plastic strains are so much larger than the elastic strain that the elastic strain many be neglected altogether. In other words, when the body is under the action of stresses below the yield point the material may be considered as rigid. Whenever this point of view be adopted the stress-strain relation of Prandtl-Reuss may be replaced by simple stress-strain relation when the elastic strain is desregarded. In this case, the total strain and plastic strain are identical. Moreover, since the plastic mean normal strain is supposed to vanish, the total mean normal strain also vanishes i.e. the material is incompressible, for

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \left(\varepsilon_x' + \varepsilon_y' + \varepsilon_z'\right) + \left(\varepsilon_x'' + \varepsilon_y'' + \varepsilon_z''\right)$$
$$= e' + e'' = e''$$

and so $e = e'' = 0$.

Thus the material is incompressible.

The stress-strain relations of Mises state that the rate of strain is proportional to stress deviator, i.e.

$$\begin{array}{l|l} \dot{\varepsilon}_x = \mu s_x, & \dot{\gamma}_{yz} = 2\mu\tau_{yz} \\ \dot{\varepsilon}_y = \mu s_y, & \dot{\gamma}_{zx} = 2\mu\tau_{zx} \\ \dot{\varepsilon}_z = \mu s_z, & \dot{\gamma}_{xy} = 2\mu\tau_{xy} \end{array} \qquad (7.43)$$

where μ is a positive factor of proportionality.

Let $I = \frac{1}{2}\left(\dot{\varepsilon}_x^2 + \dot{\varepsilon}_y^2 + \dot{\varepsilon}_z^2\right) + \frac{1}{4}\left(\dot{\gamma}_{yz}^2 + \dot{\gamma}_{zx}^2 + \dot{\gamma}_{xy}^2\right)$ (7.44)

Substituting the relations (7.43) into (7.44) we get,

$$I = \tfrac{1}{2}\mu^2\left(s_x^2 + s_y^2 + s_z^2\right) + \mu^2\left(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2\right)$$

$$= \mu^2 J_2.$$

The yield condition stipulated that

$$J_2 = k^2; \mu^2 = \frac{I}{K}.$$

Therefor, $\mu = \dfrac{\sqrt{I}}{K}$ (7.45)

Hence Von-Mises stress-strain relations are

$$\left.\begin{array}{l|l} s_x = \dfrac{K}{\sqrt{I}}\dot{\varepsilon}_x & \tau_{yz} = \dfrac{K}{2\sqrt{I}}\dot{\gamma}_{yz} \\ s_y = \dfrac{K}{\sqrt{I}}\dot{\varepsilon}_y & \tau_{zx} = \dfrac{K}{2\sqrt{I}}\dot{\gamma}_{zx} \\ s_z = \dfrac{K}{\sqrt{I}}\dot{\varepsilon}_z & \tau_{xy} = \dfrac{K}{2\sqrt{I}}\dot{\gamma}_{xy} \end{array}\right. \qquad (7.46)$$

When the strain rate is given, subject to the condition of incompressibility

$$\dot{\varepsilon}_x + \dot{\varepsilon}_y + \dot{\varepsilon}_z = 0,$$

the rates of strain be computed from (7.44) and then the stress strain relations (7.46) give the components of stress diviator.

In equation (7.46) if we consider the principal rate of strain and the principal components of stress deviator, we have.

$$\dot{\varepsilon}_1 = \mu s_1,\ \dot{\varepsilon}_2 = \mu s_2,\ \dot{\varepsilon}_3 = \mu s_3.$$

so that $\dfrac{s_1}{s_2} = \dfrac{\dot{\varepsilon}_1}{\dot{\varepsilon}_2} = \dfrac{d\varepsilon_1}{d\varepsilon_2}$

Again we have $s_1 = \sigma_1 - s = \sigma_1 - \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \dfrac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3)$

Similarly $s_2 = \dfrac{1}{3}(2\sigma_2 - \sigma_1 - \sigma_3)$, $s_3 = \dfrac{1}{3}(2\sigma_3 - \sigma_1 - \sigma_2)$

Therefore $\dfrac{d\varepsilon_1}{d\varepsilon_2} = \dfrac{2\sigma_1 - \sigma_2 - \sigma_3}{2\sigma_2 - \sigma_1 - \sigma_3}$ (7.47)

Similarly, we get two other relations of the form (7.47).

7.4 Elasto-plastic problems

1. Spherical shell under internal pressure

Let p be the internal pressure which acts uniformly on the inner boundary of a spherical shell whose internal and external radii are a and b respectively. From the central symmetry of the shell and of applied forces it can be concluded that the field of displacement vectors also posses a central symmetry. Hence all points on a concentric spherical surface of radius r have the same radially directed displacement.

Thus using sperical polar co-ordinates the displacement components are given by

$$\left(u_r, u_\theta, u_\phi\right) = \left(u(r), 0, 0\right)$$

and the nonvanishing strain components are

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \varepsilon_\phi = \frac{u}{r} \tag{7.48}$$

$$\Delta = \varepsilon_r + \varepsilon_\theta + \varepsilon_\phi = \frac{du}{dr} + \frac{2u}{r} \tag{7.49}$$

From the stress-strain relation, we have

$$\left.\begin{aligned} \sigma_r &= \lambda\Delta + 2G\varepsilon_r \\ \sigma_\theta &= \sigma_\phi = \lambda\Delta + 2G\varepsilon_\theta \end{aligned}\right\} \tag{7.50}$$

The only equation of equilibrium is

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0$$

Using (7.48), (7.49) and (7.50), the equation of equilibrium when expressed in terms of the displacement component becomes.

$$(\lambda + 2G)\frac{d}{dr}\left(\frac{du}{dr} + \frac{2u}{r}\right) = 0$$

The solution is given by

$$u = Ar + \frac{B}{r^2} \tag{7.51}$$

Then using the above relations we obtain

$$\sigma_r = (3\lambda + 2G)A - 4G\frac{B}{r^3}$$

$$\sigma_\theta = \sigma_\phi = (3\lambda + 2G)A + 2G\frac{B}{r^3}$$

The boundary conditions are

$$\sigma_r = 0 \text{ at } r = b$$

$$\sigma_r = -p \text{ at } r = a$$

Using these boundary conditions, we get

$$(3\lambda + 2G)A - 4G\frac{B}{b^3} = 0, \quad (3\lambda + 2G)A - 4G\frac{B}{a^2} = -p$$

so that $(3\lambda + 2G)A = \frac{p}{L}$, $4GB = \frac{b^3 p}{L}$, where $L = \frac{b^3}{a^3} - 1$

and hence

$$\left.\begin{aligned} \sigma_r &= -\frac{p}{L}\left(\frac{b^3}{r^3} - 1\right) \\ \sigma_\theta = \sigma_\phi &= \frac{p}{L}\left(\frac{b^3}{2r^3} + 1\right) \end{aligned}\right\} \tag{7.52}$$

Now Von Mises yield criteria is

$$(\sigma_\theta - \sigma_r)^2 + (\sigma_r - \sigma_\phi)^2 + (\sigma_\phi - \sigma_\theta)^2 = 2\sigma_0^2$$

Since, $\sigma_\theta = \sigma_\phi$, we have

$$\sigma_\theta - \sigma_r = \sigma_0 \tag{7.53}$$

Therefore, from (7.52), we have

$$\begin{aligned} \sigma_\theta - \sigma_r &= \frac{3}{2}\frac{p}{L}\frac{b^3}{r^3} \\ &= \frac{3p}{2r^3}\frac{a^3}{1 - \frac{a^3}{b^3}} \end{aligned}$$

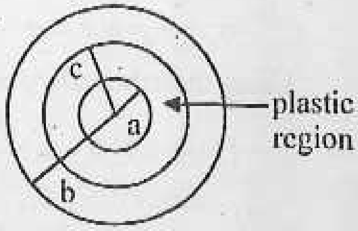
This shows that, $\sigma_\theta - \sigma_r$ is maximum when $r = a$.

Hence, yielding begins at the inner surface. The corresponding pressure p_0 is given by

$$\sigma_\theta - \sigma_r = \sigma_0 = \frac{3p_0}{2a^3}\frac{a^3}{1-\frac{a^3}{b^3}}$$

so, $$p_0 = \frac{2\sigma_0}{3}\left(1-\frac{a^3}{b^3}\right) \tag{7.54}$$

With the increase of pressure a plastic region spreads into the shell. Due to symmetry, plastic boundary in a homogeneous medium must a spherical surface. Let its radius at any instant be c. In the elastic region the equilibrium is maintained by same internal pressure which is at present unknown and the stress will be of the form.



$$\sigma_r = -A_1\left(\frac{b^3}{r^3}-1\right)$$

$$\sigma_\theta = \sigma_\phi = A_1\left(\frac{b^3}{2r^3}+1\right) \tag{7.55}$$

where A_1 is a parameter.

Now, the material just on the elastic side of plastic boundary must be on the point of yielding and so from (7.55)

$$\sigma_\theta - \sigma_r\Big|_{r=c} = \frac{3}{2}\frac{A_1 b^3}{r^3}\Big|_{r=c} = \sigma_0$$

i.e., $A_1 = \frac{2\sigma_0}{3}\frac{c^3}{b^3}$.

Therefore, for (7.55) the stress components in the elastic region $c \le r \le b$ are

$$\sigma_r = \frac{-2\sigma_0 c^3}{3b^3}\left(\frac{b^3}{r^3}-1\right) \tag{7.56}$$

$$\sigma_\theta = \sigma_\phi = \frac{2\sigma_0 c^3}{3b^3}\left(\frac{b^3}{2r^3}+1\right) \text{ on } c \le r \le b$$

Thus the solution in the elastic region is dependent only on the parameter c.

In the plastic region, the equation of equilibrium is

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0$$

or $$\frac{d\sigma_r}{dr} = \frac{2}{r}(\sigma_\theta - \sigma_r) = \frac{2\sigma_0}{r}$$

Integrating we get

$$\sigma_r = 2\sigma_0 \ln r + B_1 \tag{7.57}$$

Since σ_r is contineous across the elasto-plastic boundary $r = c$, we have,

$$2\sigma_0 \ln c + B_1 = \frac{2\sigma_0}{3}\left(1 - \frac{c^3}{b^3}\right).$$

This gives B_1.

Hence from (7.57), we get

$$\sigma_r = 2\sigma_0 \ln\frac{r}{c} - \frac{2\sigma_0}{3}\left(1 - \frac{c^3}{b^3}\right) \tag{7.58}$$

$$\sigma_\theta = \sigma_\phi = \sigma_r + \sigma_0$$

If p_1 is the pressure needed to produce plastic deformation upto radius c, then

$$p_1 = (-\sigma_r)_{r=a} = 2\sigma_0 \ln\frac{c}{a} + \frac{2\sigma_0}{3}\left(1 - \frac{c^3}{b^3}\right).$$

We know that the yielding begins when the pressure on the inner boundary is

$$p_0 = \frac{2\sigma_0}{3}\left(1 - \frac{a^3}{b^3}\right)$$

For a thin shell $b = a + t = a\left(1+\frac{t}{a}\right)$ where t is very small, so

$$p_0 = \frac{2\sigma_0}{3}\left\{1-\left(1+\frac{t}{a}\right)^{-3}\right\} = \frac{2\sigma_0 t}{a}$$

If there is sphesical cavity in an infinitely extended material we have on making $b \to \infty$.

$$p_0 = \frac{2\sigma_0}{3}.$$

i.e. the inner surface yields at a pressure which is $\frac{2}{3}$ of the yield stress.

2. Torsion of a cylindrical bar of solid circular section

Following Saint Venant we write for a torsional problem of a circular cylinder of elastic meterial, the displacement components at any point (x, y, z) in the cylinder are

$$u = -\tau yz \qquad (7.59)$$

$$v = \tau xy$$

$$w = 0$$

where τ is a constant and the axis of z is taken along the axis of cylinder.

The system of stresses associated with the displacement (7.59) is given by

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{xz} = G\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) = -G\tau y \qquad (7.60)$$

$$\tau_{yz} = G\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) = G\tau x$$

Obviously, the stresses satisfy the equations of equilibrium under no body forces and the equations of compatibility.

In polar co-ordinates

$$\tau_{rz} = \tau_{xz}\cos\theta + \tau_{yz}\sin\theta$$

$$= G\tau(-r\sin\theta\cos\theta + r\sin\theta\cos\theta) = 0$$

$$\tau_{\theta z} = \tau_{xz}\cos(90^\circ + \theta) + \tau_{yz}\cos\theta \quad (7.61)$$

$$= G\tau r \,.$$

More over

$$\tau_{xz}^2 + \tau_{yz}^2 = G^2\tau^2 r^2 = \tau_{\theta z}^2$$

The applied torque on a section is

$$Q = \text{Total Torque} = \int_0^{2\pi}\int_0^a r\tau_{\theta z} r\, dr\, d\theta$$

$$= \frac{\pi}{2} G\tau a^4 \quad (7.62)$$

Therefore

$$\frac{Q}{\tau_{\theta z}} = \frac{\pi a^4}{2r}$$

Or, $$\tau_{\theta z} = \frac{2Qr}{\pi a^4} \quad (7.63)$$

For a given torque $\tau_{\theta z}$ has its maximum value when $r = a$. Hence yielding for sufficiently large torque begins at the outer surface where $\tau_{\theta z}$ attains the greatest value.

The Von Mises condition for yielding is

$$J_2 = \frac{1}{2}\left(s_x^2 + s_y^2 + s_z^2\right) + \tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2 = \frac{\sigma_0^2}{3}$$

In the persent case, we have

$$\tau_{xz}^2 + \tau_{yz}^2 = \frac{\sigma_0^2}{3}$$

$$\tau_{\theta z}^2 = \frac{\sigma_0^2}{3}$$

Therefore $$\tau_{\theta z} = \frac{\sigma_0}{\sqrt{3}} \quad (7.64)$$

Hence if an element at a distance r from the centre is in a plastic state, then

$$\frac{\sigma_0}{\sqrt{3}} = \left(\tau_{\theta z}\right)_{\text{at } r} = \frac{2Qr}{\pi a^4}$$

$$\therefore Q = \frac{\pi a^4 \sigma_0}{2\sqrt{3}r} \qquad (7.65)$$

Suppose that Q_i is the value of the torque when yelding just begins at $r = a$.

Then $Q_i = \dfrac{\pi a^3 \sigma_0}{2\sqrt{3}}$.

Again from (7.65) we have,

$$r = \frac{\pi a^4 \sigma_0}{2\sqrt{3}Q}.$$

Hence as Q increases the plastic deformation occurs for smaller values of r.

For $Q > Q_i$ a plastic zone will develop between the outer surface and a concentric cylindrical surface. The radius ρ of the plastic front or elasto-plastic boundary depends on the magnitude of the applied torque. In the plastic zone we have

$$\tau_{\theta z} = \frac{\sigma_0}{\sqrt{3}}$$

and in the elastic zone

$$\tau_{\theta z} = G\tau r.$$

So at the elastic plastic boundary $r = \rho$, from the continuity of the stresses, we have

$$G\tau\rho = \frac{\sigma_0}{\sqrt{3}}$$

$$\rho = \frac{\sigma_0}{\sqrt{3}G\tau} \qquad (7.66)$$

Now, the torque corresponding to a given value of ρ is

$$Q = \int_0^{2\pi}\int_0^a r\tau_{\theta z} r\,d\theta\,dr$$

$$= 2\pi\int_0^{\rho} G\tau r.r^2 dr + 2\pi\int_{\rho}^{a}\frac{\sigma_0}{\sqrt{3}}r^2 dr$$

$$= \frac{2\pi\sigma_0 a^3}{3\sqrt{3}} - \frac{\pi\sigma_0^4}{54G^3\tau^3} \qquad (7.67)$$

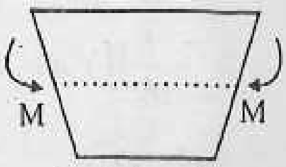
where equation (7.66) has been used.

If the whole cylinder had been plastic the corresponding torque will be

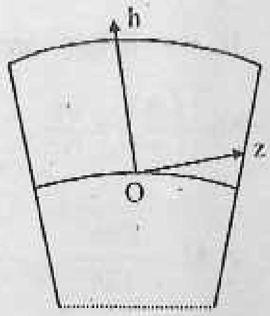
$$Q_l = 2\pi\int_0^{a}\frac{\sigma_0}{\sqrt{3}}r^2\,dr = \frac{2\pi\sigma_0 a^3}{3\sqrt{3}} \qquad (7.68)$$

From (7.67) it is evident that Q_l of Q can be obtained when τ is infinitely large or when $\tau_{\theta z} = G\tau r$ is infinite, which is physically not possible.

Thus we find that the cylinder cannot be fully plasitc. There is always a core of material which is elastic.

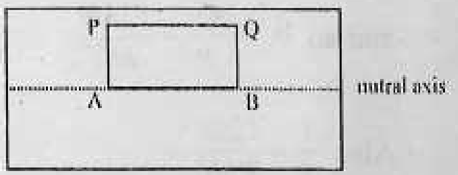


Central section

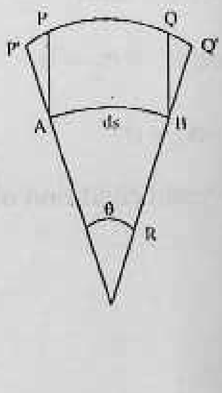


3. Bending of a prismatic bar for a narrow rectangular cross-section by terminal couples.

In each longitudinal section, the upper fibre is stretched while the lower fibre is compressed. Thus there must be a line in this section which remains unstretched. This line is called the *nuetral line*. The axis of z is along the neutral line of the central section which is supposed to be bent to arc of constant curvature $\frac{1}{R}$. The axis of y is perpendicular to the z axis and is in the plane of bending as shown is figure.



If a fibre PQ of length ds a a distance y from neutral is extended to the length $P'Q'=ds'$, then



$$\text{extension } \varepsilon = \frac{P'Q' - PQ}{PQ}$$

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

Hence

$$\sigma_z = E.\varepsilon = \frac{Ey}{R} \tag{7.69}$$

We observe that y is positive, when σ is positive and when y is negative. σ_z becomes negative.

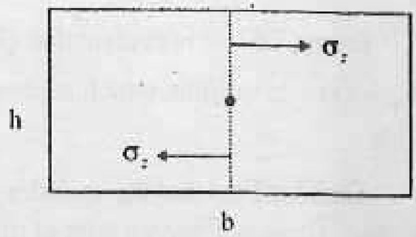
Now, M = moment of traction across any section = bending moment.

$$= \iint y\sigma_z dA$$

$$= \iint \frac{Ey^2}{R} dA$$

$$= \frac{E}{R}\iint y^2 dA$$

$$= \frac{E\,I}{R}$$



If the height and breadth of the section be h and b respectively, then

$$I = bh\frac{1}{3}\left(\frac{h}{2}\right)^2 = \frac{bh^3}{12}$$

$$\therefore M = \frac{Ebh^3}{12h} \tag{7.70}$$

and so $\sigma_z = \frac{Ey}{R} = \frac{12My}{bh^3}$ (7.71)

Also $\frac{1}{R} = \frac{12M}{Ebh^3}$ (7.72)

Since $\sigma_1 = \sigma_2 = 0$

and $\sigma_3 = \sigma_z$

The yield condition of Mises is

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_0^2$$

$$\sigma_3=\sigma_z=\pm\sigma_0 \tag{7.73}$$

From (7.69) it is evident that the yielding begins when y has got mumerically greatest value i.e., when $y=\pm\frac{h}{2}$.

Suppose M_i is the corresponding bending moment. Therefore, from (7.71), we have when $y=\frac{h}{2}$.

$$\sigma_z=\sigma_0=\frac{12M_i\,{}^{a}\!/_{2}}{bh^3}=\frac{6M_i}{bh^2}$$

and when $y=-\frac{h}{2}$

$$\sigma_z=-\sigma_0=-\frac{6M_i}{bh^2}$$

Or $$M_i=\frac{\sigma_0 bh^2}{6} \tag{7.74}$$

The curvature of the neutral line is obtained from (7.72) as

$$\frac{1}{R}=\frac{12M_i}{Ebh^3}=\frac{2\sigma_0}{Eh} \tag{7.75}$$

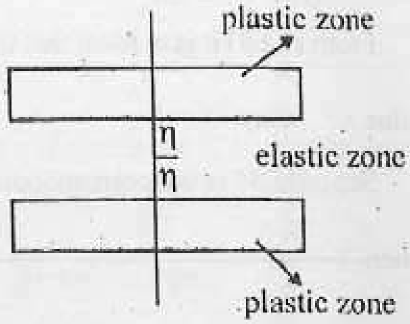
If there be a plastic element at a distance y from the neutral line when the bending moment is M, then we have from (7.71)

$$\sigma_z=\sigma_0=\frac{12My}{bh^3}$$

Or $$y=\frac{\sigma_0 bh^3}{12M}$$

This relation shows that y decreases with the increase of M provided $M>M_i$. If the central position of the beam $-\eta\leq y\leq\eta$ is an elastic state, then

$$\sigma_z = \frac{Ey}{R} \quad -\eta < y < \eta$$
$$= \sigma_0 \quad |y| > \eta$$



At the elasto-plastic boundary $y = \eta$ we have

$$\frac{E\eta}{R} = \sigma_0 \quad \text{Or,} \quad \frac{E}{R} = \frac{\sigma_0}{\eta}.$$

Hence

$$\sigma_z = \frac{\sigma_0 y}{R}, \quad -\eta \le y \le \eta$$
$$= \sigma_0 \quad , |y| > \eta$$

The bending moment corresponding to an assumed value of η is

$$M = \int_{-h/2}^{h/2} y\sigma_z b\,dy$$

$$= 2b\int_0^{h/2} y\sigma_z dy \quad [y\sigma_z \text{ is even}]$$

$$= 2b\int_0^{\eta} y\frac{\sigma_0 y}{\eta}dy \; + \; 2b\int_{\eta}^{h/2} y\sigma_0 dy$$

↑ elastic   ↑ plastic

$$= b\sigma_0\frac{h^2}{4} - b\sigma_0\frac{\eta^2}{3}$$

Using the result $\sigma_0 = \frac{E\eta}{R}$, we have

$$M = \frac{b\sigma_0 h^2}{4} - \frac{1}{3}b\sigma_0^3\frac{R^2}{E^2} \qquad (7.76)$$

Now, if the whole section had been plastic we obtain

M_f= bending moment required for full plastic defomation

$$= 2b\int_0^{h/2} y\sigma_z dy = 2b\int_0^{h/2} y\sigma_0 dy$$

$$= \frac{b\sigma_0 h^2}{4} \qquad (7.77)$$

Compairing (7.77) with (7.76) we find that full plastic deformation requires $R = 0$ and $\frac{1}{R}$ is infinitely large. This means that $\sigma_z = \frac{Ey}{R}$ is infinitely large which is physically not possible. Hence the bar cannot be mad fully plastic. There must be an elastic core.

7.5 Summary

In this unit the basic relationships of the theory of plasticity have been discurred. Yield criterion, viz. Trisca's Criterion and Von Mises Criterion are also discussed. The equations of plasticity of Prandtl Reuss theory and stress-strain relations of Von-Mises have been derived. Some Elasto-plastic problems are also considered.

7.6 Exercises

1. Short Answer Type :

(a) Define yield condition.

(b) Explain Tresca's Criterion.

(c) What are the stress and strain deviators.

(d) Show that the component of the stress deviators can be considered as components of stress involving no dilatation.

2. Broad Answer Type :

(a) Derive stress-strain relations of Von-Mises. Show that for incompressible strain rate the stress-strain relations are expressed in terms of the components of stress deviator.

(b) A thick walled spherical shell of internal radius a and external radius b is subjected to a uniform pressure p on its inner surface. Show that there will be a plastic zone bounded by $r = a$ and $r = c$, $a \leq c \leq b$ provided that p is given by

$$p = 2\sigma_0 \ln\frac{c}{a} + \frac{2}{3}\sigma_0\left(1 - \frac{c^3}{b^3}\right)$$

Deduce that for a thin shell of thickness t, the pressure p_0 needed to being yielding is

$$p_0 = \frac{2\sigma_0 t}{a},\ \sigma_0 \text{ being the yield constant for the material.}$$

(c) Deduce the Prandtl-Ruess stress-strain relations for plastic flow in an elasto-plastic medium in the form.

$$\dot{s}_x = 2G\left(\dot{e}_x - \frac{\dot{W}}{2k^2}s_x\right) \text{ and two similar equation}$$

$$\dot{\tau}_{yz} = G\left(\dot{\gamma}_{yz} - \frac{\dot{W}}{k^2}\tau_{yz}\right) \text{ and two similar equations.}$$

(d) A prismatic bar of narrow rectangular cross-section is bent by terminal couples such that the bar is partly elastic and partly plastic. Find by using Von-Mises Criterion, the moment of the couple and show that the bar cannot be fully plastic.

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মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

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